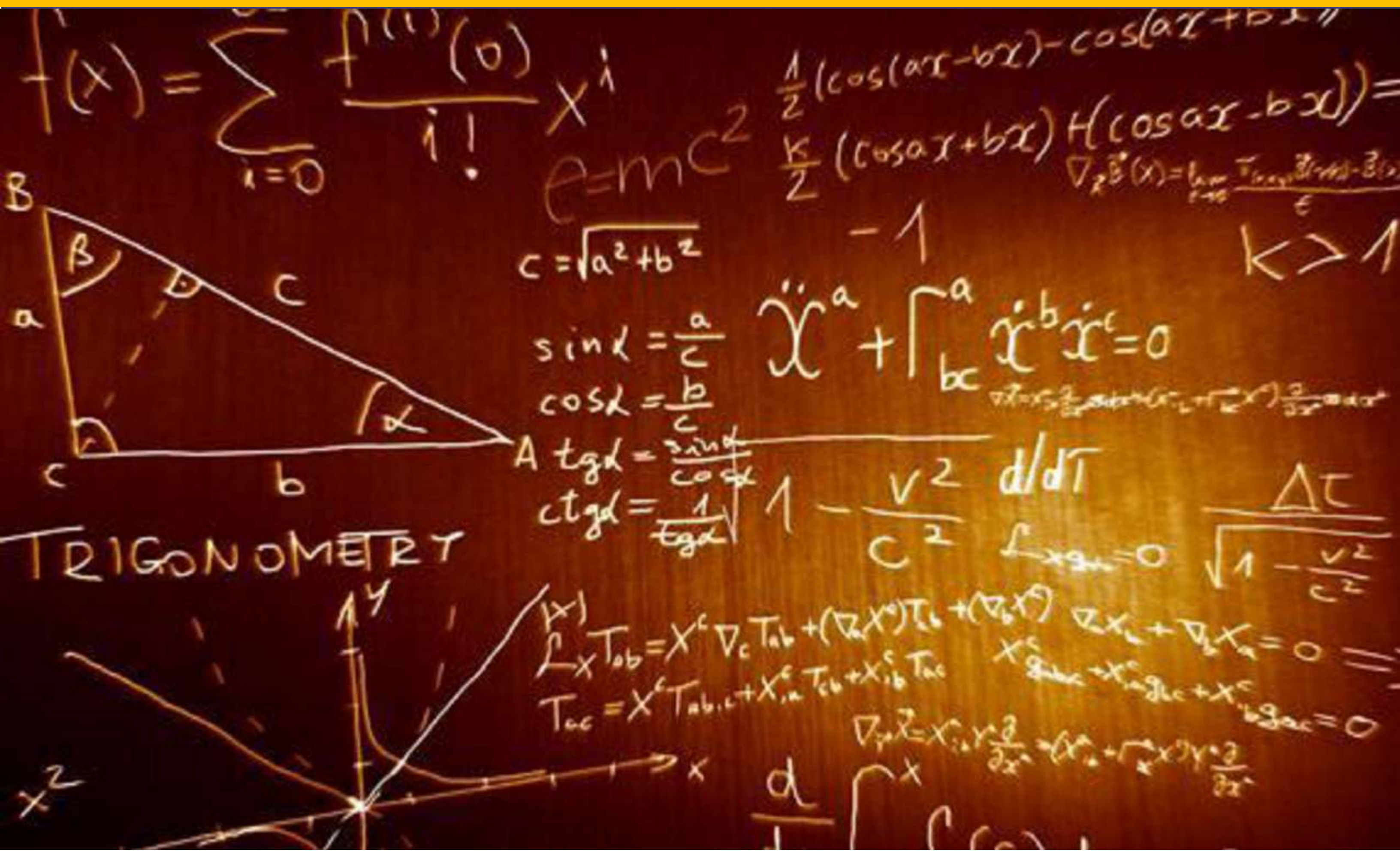
Advances in

Mathematical Sciences

(A Collection of Survey Research Articles)

Edited By
Dr. Zakir Ahmed



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Preface

The present book is a collection of several survey articles written by people from Assam, describing mainly their work. The topics selected are from various areas of mathematics and statistics. The book begins with five articles in algebra enunciating the recent developments in those areas. Then we have five articles in number theory with topics ranging from congruences to partitions and finally to Pell's equations. The rest of the articles present some advances in game theory, topology, functional analysis and statistics.

It is hoped that this book would serve as a ready reference for someone who is interested in the topics presented here. A generous sprinkling of open problems in almost all the articles makes it easy to look for research problems in these areas and the editor hopes that it will serve the mathematical community well.

Dr. Zakir Ahmed

Contents

Some results on commuting probability, n-centralizer rings and non-commuting graph of finite rings

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Abstract. The commuting probability of a finite ring R, denoted by $\Pr(R)$, is the probability that a randomly chosen pair of elements of R commute. In this chapter, we describe some results on commuting probability and its relations with n-centralizer rings and non-commuting graphs of finite rings.

2010 Mathematical Sciences Classification. 16U70.

Keywords. finite ring, centralizer, commuting probability.

1 Introduction

Let F be an algebraic system having finite number of elements which is closed under a multiplication operation. The commuting probability of F, denoted by $\Pr(F)$, is the probability that a randomly chosen pair of elements of F commute. That is

$$\Pr(F) = \frac{|\{(x, y) \in F \times F : xy = yx\}|}{|F \times F|}.$$

Note that $\Pr(F) = 1$ if and only if F is commutative.

Many papers have been written on commuting probability of finite groups in the last few decades, for example see [9, 19, 20, 21, 22, 23, 25, 27] etc., starting from the works of Erdős and Turán [ETa68].

The study of commuting probability of a finite ring was neglected over the years. At this moment, we have only a handful of papers on this topic including [24] where MacHale initiated the study of commuting probability of a finite ring. In the year 1976, MacHale [24] showed that for any finite ring R, $\Pr(R) \notin (\frac{5}{8}, 1)$. After many years, in the year 2013, MacHale resumed the study of commuting probability of finite rings together with Buckley and Shé (see [7, 8]). Buckley, MacHale and Shé [8] also introduced the concept of $\mathbb{Z}$-isoclinism between two finite rings.

Throughout this article R denotes a finite ring and $Z(R)$ denotes the center of R. That is, $Z(R) = \{x \in R : xy = yx \text{ for all } y \in R\}$. For any two elements $r, t \in R$, we write $[r, t]$ to denote the additive commutator of r and t. That is, $[r, t] = rt - tr$. Also, for any $x \in R$, we write $[x, R]$ to denote the subgroup of $(R, +)$ consisting of elements of the form $[x, y]$ where $y \in R$ and $[R, R]$ to denote the subgroup of $(R, +)$ consisting of elements of the form $[r, t]$ where $r, t \in R$. Again, we write $\frac{R}{S}$ to denote the additive quotient group, for any subring S of R, and $|R : S|$ to denote the index of $(S, +)$ in $(R, +)$. Further, if S is an ideal of R then we also write $\frac{R}{S}$ to denote the quotient ring. The isomorphisms considered are the additive group isomorphisms. Two rings R_1 and R_2 are said to be $\mathbb{Z}$-isoclinic if there exist additive group isomorphisms $\phi : \frac{R_1}{Z(R_1)} \to \frac{R_2}{Z(R_2)}$ and $\psi : [R_1, R_1] \to [R_2, R_2]$ such that $\psi([u, v]) = [u', v']$ whenever $\phi(u + Z(R_1)) = u' + Z(R_2)$ and $\phi(v + Z(R_1)) = v' + Z(R_2)$. Equivalently, the following diagram commutes

$$\begin{array}{ccc}
\frac{R_1}{Z(R_1)} \otimes \frac{R_1}{Z(R_1)} & \xrightarrow{\phi\otimes\phi} & \frac{R_2}{Z(R_2)} \otimes \frac{R_2}{Z(R_2)} \\
\downarrow{\scriptstyle a_{R_1}} & & \downarrow{\scriptstyle a_{R_2}} \\
[R_1, R_1] & \xrightarrow{\psi} & [R_2, R_2]
\end{array}$$

where $\frac{R_i}{Z(R_i)} \otimes \frac{R_i}{Z(R_i)}$ denotes tensor product of $\frac{R_i}{Z(R_i)}$ with itself for $i = 1, 2$;
$a_{R_i} : \frac{R_i}{Z(R_i)} \otimes \frac{R_i}{Z(R_i)} \to [R_i, R_i],\ i = 1, 2$ are well defined maps given by

$$a_{R_i}((x_i + Z(R_i)) \otimes (y_i + Z(R_i))) = [x_i, y_i]$$

for all $x_i, y_i \in R_i$ and $i = 1, 2$; and

$$(\phi \otimes \phi)((x_1 + Z(R_1)) \otimes (y_1 + Z(R_1))) = (x_2 + Z(R_2)) \otimes (y_2 + Z(R_2))$$

whenever $\phi(x_1 + Z(R_1)) = x_2 + Z(R_2)$ and $\phi(y_1 + Z(R_1)) = y_2 + Z(R_2)$. The above diagram commutes means

$$a_{R_2} \circ (\phi \otimes \phi) = \psi \circ a_{R_1}.$$

The pair of mappings (ϕ, ψ) is called a $\mathbb{Z}$-isoclinism between R_1 and R_2. Buckley, MacHale and Níshé [8] showed that the commuting probabilities of two finite $\mathbb{Z}$-isoclinic rings are same.

Let $\mathrm{Cent}(F) = \{C_F(x) : x \in F\}$, where $C_F(x) = \{y \in F : xy = yx\}$. F is called n-centralizer if $|\mathrm{Cent}(F)| = n$. The study of finite n-centralizer group was initiated by Belcastro and Sherman [6], in the year 1994. Following them, many Mathematicians have studied finite n-centralizer groups in the recent years (see for example [1, 2, 3, 4, 5, 6, 10] etc.).

The non-commuting graph of a finite ring R, denoted as $\Gamma(R)$, is a graph whose vertex set is $R \setminus Z(R)$ and there is an edge between two vertices a and b if and only if $ab \neq ba$. The notion of non-commuting graph of a finite ring was introduced by Erfanian, Khashyarmanesh and Nafar [18]. It is worth mentioning that commuting graph of finite group was originated from a problem on groups posed by Paul Erdős [26]. In this chapter, we describe some results on commuting probability and its relations with n-centralizer rings and non-commuting graphs of finite rings.

Let $\mathcal{G}$ and $\mathcal{H}$ be any two graphs. We write $V(\mathcal{G})$ and $E(\mathcal{G})$ to denote the set of vertices and the set of edges of $\mathcal{G}$ respectively. Let $d(x, y)$ be the length of the shortest path from the vertices x to y. Then the diameter of $\mathcal{G}$, denoted by $\mathrm{diam}(\mathcal{G})$, is given by $\max\{d(x, y) : x, y \in V(\mathcal{G})\}$. The girth of $\mathcal{G}$, denoted by $\mathrm{girth}(\mathcal{G})$, is the length of the shortest cycle in $\mathcal{G}$. We write $\deg(v)$ to denote the degree of a vertex v, which is the number of edges incident on v. A dominating set of a graph $\mathcal{G}$ is a subset D of $V(\mathcal{G})$ such that every vertex in $V(\mathcal{G}) \setminus D$ is adjacent to at least one member of D. A bijective map $f : V(\mathcal{G}) \to V(\mathcal{H})$ is called an isomorphism between the graphs $\mathcal{G}$ and $\mathcal{H}$ if any two vertices $u, v \in V(\mathcal{G})$ are adjacent if and only if $f(u), f(v) \in V(\mathcal{H})$ are adjacent. If there exists an isomorphism between two graphs then the graphs are said to be isomorphic.

2 Bounds for commuting probability of finite rings

We begin this section with the following three central problems in the study of commuting probability of a finite ring R.

(i) Which real numbers belonging to $(0, 1]$ can be realized as commuting probability for some finite ring. In other words, if $\mathcal{R}$ denotes the set of all finite rings then determine the set $\{\Pr(R) : R \in \mathcal{R}\}$.

(ii) Is it possible to characterize finite rings in terms of its commuting probability?

(iii) Is it possible to obtain some bounds for $\Pr(R)$ in terms of some well-known ring-theoretic notions?

In 1976, MacHale [24] proved the following results related to the above problems.

Theorem 2.1. *[24, Theorem 1] If R is a finite non-commutative ring then $\Pr(R) \leq \frac{5}{8}$. The equality holds if and only if $|R : Z(R)| = 4$.*

Above theorem shows that there is no finite ring R such that $\Pr(R) \in (\frac{5}{8}, 1)$.

Theorem 2.2. *[24, Theorem 2] Let R be a non-commutative ring and p the smallest prime dividing order of R. Then*

$$\Pr(R) \leq \frac{p^2 + p - 1}{p^3}.$$

The equality holds if and only if $|R : Z(R)| = p^2$.

Theorem 2.3. *[24, Theorem 4] If S is a subring of a finite ring R then $\Pr(R) \leq \Pr(S)$.*

In [11], we have obtained the following bounds for $\Pr(R)$.

Theorem 2.4. *[11, Theorem 2.1(a)] Let R be a finite non-commutative ring. If p is the smallest prime dividing $|R|$ then*

$$\Pr(R) \geq \frac{|Z(R)|}{|R|} + \frac{p(|R| - |Z(R)|)}{|R|^2}$$

with equality if and only if $|C_R(r)| = p$ for all $r \notin Z(R)$.

Let $K(R, R)$ denotes the set $\{[s, r] : s \in R, r \in R\}$ and $[R, R]$ denotes the subgroup of $(R, +)$ generated by $K(R, R)$. Then we have the following bounds.

Theorem 2.5. *[11, Theorem 2.4] Let R be a finite ring. Then*

$$\Pr(R) \geq \frac{1}{|K(R,R)|}\left(1 + \frac{|K(R,R)| - 1}{|R : Z(R)|}\right)$$

with equality if and only if $|K(R,R)| = |[r,R]|$ for all $r \in R \setminus Z(R)$. In particular, if R is non-commutative then $\Pr(R) > \frac{1}{|K(R,R)|}$.

Theorem 2.6. *[11, Theorem 2.5] Let R be a finite ring. Then*

$$\Pr(R) \geq \frac{1}{|[R,R]|}\left(1 + \frac{|[R,R]| - 1}{|R : Z(R)|}\right)$$

with equality if and only if $|[R,R]| = |[r,R]|$ for all $r \in R \setminus Z(R)$. In particular, if R is non-commutative then $\Pr(R) > \frac{1}{|[R,R]|}$.

We observe that the lower bound obtained in Theorem 2.5 is better than the lower bounds obtained in Theorem 2.4 and Theorem 2.6.

We have also obtained the following upper bounds.

Theorem 2.7. *[11, Theorem 2.1(b)] Let R be a finite non-commutative ring. If p is the smallest prime dividing $|R|$ then*

$$\Pr(R) \leq \frac{(p-1)|Z(R)| + |R|}{p|R|}$$

with equality if and only if $|R : C_R(r)| = p$ for all $r \notin Z(R)$.

Theorem 2.8. *[11, Theorem 2.3] Let N be an ideal of a finite non-commutative ring R. Then*

$$\Pr(R) \leq \Pr(R/N)\Pr(N).$$

The equality holds if $N \cap [R,R] = \{0\}$.

We observe that the upper bound obtained in Theorem 2.7 is better than Theorem 2.2 and Theorem 2.8 is an improvement of Theorem 2.3.

3 Commuting probability and n-centralizer finite rings

A ring R is called an n-centralizer ring if $|\operatorname{Cent}(R)| = n$. In [13], we introduce and study n-centralizer finite rings. Some of the results are given below.

Proposition 3.1. *[13, Proposition 2.1] R is a commutative ring if and only if R is 1-centralizer ring.*

Theorem 3.2. *[13, Theorem 2.6] For any non-commutative ring R,*

$$|\operatorname{Cent}(R)| \geq 4.$$

Theorem 3.3. *[13, Theorem 2.12] Let R be a non-commutative ring whose order is a power of a prime p. Then $|Cent(R)| \geq p + 2$, and equality holds if and only if $\frac{R}{Z(R)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$.*

Theorem 3.4. *[13, Theorem 3.2] Let R be a non-commutative finite ring. Then R is a 4-centralizer ring if and only if $\frac{R}{Z(R)} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.*

Theorem 3.5. *[13, Theorem 4.3] Let R be a finite ring. Then R is a 5-centralizer ring if and only if $\frac{R}{Z(R)} \cong \mathbb{Z}_3 \times \mathbb{Z}_3$.*

In [16], we have also obtained the following results.

Theorem 3.6. *[16, Theorem 3.3] If R is a 6-centralizer finite ring then*

$$\frac{R}{Z(R)} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_6, \mathbb{Z}_2 \times \mathbb{Z}_8, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4 \text{ or } \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2.$$

Theorem 3.7. *[16, Theorem 3.5] Let R be a finite 7-centralizer ring. Then*

$$\frac{R}{Z(R)} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_6 \text{ or } \mathbb{Z}_5 \times \mathbb{Z}_5.$$

The following theorems give commuting probabilities of some finite n-centralizer rings.

Theorem 3.8. *[13, Theorem 5.4] and [16, Theorem 2.2] Let R be a non-commutative ring whose order is a power of a prime p. Then R is $(p+2)$-centralizer ring if and only if $\Pr(R) = \frac{p^2+p-1}{p^3}$.*

Theorem 3.9. *[16, Theorem 2.3(a)] Let R be a finite ring. Then R is 4-centralizer if and only if $\Pr(R) = \frac{5}{8}$.*

Theorem 3.10. *[16, Theorem 2.3(b)] If R is a finite 5-centralizer ring then $\Pr(R) = \frac{11}{27}$.*

Theorem 3.11. *[16, Theorem 3.4] If R is a 6-centralizer finite ring such that $\frac{R}{Z(R)}$ is not isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ then $\Pr(R) \in \{\frac{7}{16}, \frac{35}{72}, \frac{29}{64}\}$.*

Theorem 3.12. *[16, Theorem 3.6] Let R be a finite 7-centralizer ring such that $\frac{R}{Z(R)}$ is not isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_6$. Then $\Pr(R) = \frac{29}{125}$.*

4 Commuting probability and non-commuting graph of rings

The notion of non-commuting graph of a finite ring was introduced and studied by Erfanian, Khashyarmanesh and Nafar [18] recently. They have obtained the following results.

Theorem 4.1. *[18, Theorem 2.1] Let R be a finite non-commutative ring. Then $\operatorname{diam}(\Gamma_R) \leq 2$ and $\operatorname{girth}(\Gamma_R) = 3$.*

Theorem 4.2. *[18, Theorem 2.2] Let R be a finite non-commutative ring. Then Γ_R is complete if and only if $|R| = 4$.*

Theorem 4.3. *[18, Proposition 2.5] Let R be a finite non-commutative ring. Then a subset S of $V(\Gamma_R)$ is a dominating set of Γ_R if and only if $C_R(S) \subseteq Z(R) \cup S$.*

Some of the results, obtained in [12], are as follows.

Proposition 4.4. *[12, Proposition 2.1] Let R be a finite ring. Then*

1. Γ_R *is connected.*

2. Γ_R *is empty graph if and only if* R *is commutative.*

Theorem 4.5. *[12, Theorem 2.2, Theorem 2.4] Let* R *be a finite non-commutative ring. Then*

1. Γ_R *is not a star graph or a lollipop graph or a bipartite graph.*

2. Γ_R *is not a complete graph for any finite non-commutative ring* R *with unity.*

Theorem 4.6. *[12, Corollary 2.6] Let* R *be a finite non-commutative ring with unity and* $S = \{s_1, s_2, \ldots, s_n\}$ *a generating set for* R*. If* $S \cap Z(R) = \{s_{m+1}, \ldots, s_n\}$ *then* $D = \{s_1, s_2, \ldots, s_m\}$ *is a dominating set for* Γ_R*.*

The following two results give relations between Γ_R and $\Pr(R)$.

Theorem 4.7. *[12, Theorem 3.1] Let* R *be a finite non-commutative ring. Then the number of edges of* Γ_R *is*

$$|E(\Gamma_R)| = \frac{|R|^2}{2}(1 - \Pr(R)).$$

Theorem 4.8. *[12, Theorem 4.1] Let* R_1 *and* R_2 *be finite rings. Then* $\Pr(R_1) = \Pr(R_2)$ *if* $|Z(R_1)| = |Z(R_2)|$ *and* $\Gamma_{R_1} \cong \Gamma_{R_2}$*.*

We conclude this chapter with the following research problems.

Problem 4.9. *Does there exist an* n*-centralizer finite ring for any positive integer* $n \neq 2, 3$*?*

Problem 4.10. *Can we characterize* n*-centralizer finite rings for* $n \geq 8$ *(if exists)?*

Problem 4.11. *If* R *is a* 6*-centralizer finite ring such that* $\frac{R}{Z(R)}$ *is isomorphic to* $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$ *or* $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ *then determine the values of* $\Pr(R)$*.*

Problem 4.12. *If* R *is a* 7*-centralizer finite ring such that* $\frac{R}{Z(R)}$ *is isomorphic to* $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_6$ *then determine the value of* $\Pr(R)$*.*

Problem 4.13. *If* R *is an* n*-centralizer finite ring such that* $n \geq 8$ *then determine the values of* $\Pr(R)$*.*

Problem 4.14. *Determine the graphs* $\mathcal{G}$ *such that* $\mathcal{G}$ *is isomorphic to* Γ_R *for some finite ring* R*.*

5 Bibliography

[1] Abdollahi, A., Amiri, S. M. J., and Hassanabadi, A. M. Groups with specific number of centralizers. *Houston J. Math.*, 33(1):43–57, 2007.

[2] Ashrafi, A. R. Counting the centralizers of some finite groups. *Korean J. Comput. Appl. Math*, 7(1):115–124, 2000.

[3] Ashrafi, A. R. On finite groups with given number of centralizers. *Algebra Colloq.*, 7(2):139–146, 2000.

[4] Ashrafi, A. R. and Taeri, B. On finite groups with exactly seven element centralizers. *J. Appl. Math. Comput.*, 22(1–2):403–410, 2006.

[5] Baishya, S. J. On finite groups with specific number of centralizers. *Int. Electron. J. Algebra*, 13:53–62, 2013.

[6] Belcastro, S. M. and Sherman, G. J. Counting centralizers in finite groups. *Math. Magazine*, 67(5):366–374, 1994.

[7] Buckley, S. M. and MacHale, D. Contrasting the commuting probabilities of groups and rings, Retrieved on 18 January 2015 from `http://archive.maths.nuim.ie/staff/sbuckley/Papers`.

[8] Buckley, S. M., MacHale, D., and Níshé, A. Finite rings with many commuting pairs of elements, Retrieved on 18 January 2015 from `http://archive.maths.nuim.ie/staff/sbuckley/Papers/bms.pdf`.

[9] Das, A. K. and Nath, R. K. A characterization of certain finite groups of odd order. *Math. Proc. Royal Irish Acad.*, III(A2):69–78, 2011.

[10] Dutta, J. A characterization of 4-centralizer groups. *Chinese Journal of Mathematics*, 2013:871072-1–871072-2, 2013.

[11] Dutta, J. and Basnet, D. K. Some bounds for commuting probability of finite rings. *Proceedings–Mathematical Sciences*, 129(1), Article ID 1, 1–6, 2019.

[12] Dutta, J. and Basnet, D. K. On non-commuting graph of a finite ring, communicated, available at https://arxiv.org/pdf/1703.05039.pdf.

[13] Dutta, J., Basnet, D. K., and Nath, R. K. Characterizing some rings of finite order, communicated, available at https://arxiv.org/pdf/1510.01207.pdf.

[14] Dutta, J., Basnet, D. K., and Nath, R. K. A note on n-centralizer finite rings. *An. Stiint. Univ. Al. I. Cuza Iasi Mat. (N.S.)*, LXIV(f.1), 161–171, 2018.

[15] Dutta, J., Basnet, D. K., and Nath, R. K. On commuting probability of finite rings. *Indag. Math.*, 28(2):372–382, 2017.

[16] Dutta, J., Basnet, D. K., and Nath, R. K. Commuting probabilities on n-centralizer finite rings, communicated, available at https://arxiv.org/abs/1803.04111.

[17] Erdős, P. and Turán, P. On some problems of a statistical group-theory. IV. *Acta. Math. Acad. Sci. Hungar*, 19(3–4):413–435, 1968.

[18] Erfanian, A., Khashyarmanesh, K., and Nafar, Kh. Non-commuting graphs of rings. *Discrete Math. Algorithms Appl.*, 7(3):1550027-1–1550027-7, 2015.

[19] Guralnick, R. M. and Robinson, G. R. On the commuting probability in finite groups. *J. Algebra*, 300(2):509–528, 2006.

[20] Hegarty, P. Limit points in the range of the commuting probability function on finite groups. *J. Group Theory*, 16(2):235–247, 2013.

[21] Lescot, P. Isoclinism classes and commutativity degrees of finite groups. *J. Algebra*, 177:847–869, 1995.

[22] Lescot, P. Central extensions and commutativity degree. *Comm. Algebra*, 29(10):4451–4460, 2001.

[23] MacHale, D. How commutative can a non-commutative group be?. *Math. Gazette*, 58(405):199–202, 1974.

[24] MacHale, D. Commutativity in finite rings. *Amer. Math. Monthly*, 83(1):30–32, 1976.

[25] Nath, R. K. and Das, A. K. On a lower bound of commutativity degree. *Rend. Circ. Mat. Palermo*, 59(1):137–142, 2010.

[26] Neumann B. H. (1976) A problem of Paul Erdős on groups, *J. Aust. Math. Soc.*, **21**, 467–472.

[27] Rusin, D. J. What is the probability that two elements of a finite group commute?. *Pacific J. Math.*, 82(1):237–247, 1979.

A survey on the autocommuting probability of a finite group

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Abstract. For many years people have been studying the commuting probability of an algebraic structure. It is a very important notion in Algebra as it measures how much commutative an algebraic structures is. Besides it is a tool to characterizing groups and rings. For the past few decades people have been studying this notion and its generalizations. In this article, we give a brief survey on a generalization of the commuting probability of a finite group, which is called autocommuting probability of the group and collect recent results on this probability.

2010 Mathematical Sciences Classification. Primary 20D60, 20P05, 20F28.

Keywords. Automorphism group, Autocommuting probability, Autoisoclinism.

1 Introduction

Let G be a group having a finite number of elements. The commuting probability of G, denoted by $\Pr(G)$, is the probability that a randomly chosen pair of elements of G commute. That is

$$\Pr(G) = \frac{|\{(x, y) \in G \times G : xy = yx\}|}{|G \times G|}.$$

Clearly, $\Pr(G) = 1$ if and only if G is commutative. In 1968, Erdős and Turán [ETa68] initiated the study of commuting probability of finite groups. After Erdős and Turán many authors have worked on $\Pr(G)$ and its generalizations, for examples, see [DN10, EL07, RL14, NY15, ND10, PS08]. A survey on the generalizations of $\Pr(G)$ can be found in [DP13]. Sherman [She75] proposed a new direction to the generalizations of $\Pr(G)$. He considered the automorphism group of G and introduced the probability that an automorphism of a group fixes an random element of the group. We call this notion as autocommuting probability of a finite group. In this article, we discuss various results obtained for autocommuting probability including some computing formulae, some bounds and invariance property. We also give a brief survey on certain characterizations of a group in terms of autocommuting probability.

2 Autocommuting probability of a finite group

Let a finite group $\mathcal{G}$ acts on a set Ω. Sherman [She75], in the year 1975, introduced the probability (denoted by $\Pr(\Omega, \mathcal{G})$) that a randomly chosen element of $\mathcal{G}$ fixes a randomly chosen element of Ω. Note that if $\Omega = G$ and $\mathcal{G} = \operatorname{Aut}(G)$, the automorphism group of G, then $\Pr(G, \operatorname{Aut}(G))$ is nothing but the probability that an automorphism of G fixes a random element of G. Thus

$$\Pr(G, \operatorname{Aut}(G)) = \frac{|\{(x, \alpha) \in G \times \operatorname{Aut}(G) : \alpha(x) = x\}|}{|G||\operatorname{Aut}(G)|}$$

$\Pr(G, \operatorname{Aut}(G))$ is called autocommuting probability of G. Sherman studied this notion for considering finite abelian groups. He proved that

$$\Pr(G, \operatorname{Aut}(G)) = \begin{cases} 1 & \text{if } G \cong \mathbb{Z}_2 \\ \frac{3}{4} & \text{if } G \ncong \mathbb{Z}_2. \end{cases}$$

Further, Sherman [She75] proved the following result.

Theorem 2.1. *[She75, Proposition 1] If p is a prime and G is an abelian group of order p^n then*

$$\Pr(G, \operatorname{Aut}(G)) \leq 2\left(\frac{3}{p^2}\right)^{\frac{n}{2}}.$$

The study of autocommuting probability remain neglected for many years until Arora and Karan [AK17] resume the study. They have computed the values of $\Pr(G, \operatorname{Aut}(G))$ for some classes of groups and characterize G for some values of $\Pr(G, \operatorname{Aut}(G))$. Some of their results are as follows.

Theorem 2.2. *[AK17, Theorem 5] Let G be a finite abelian group. Then $\Pr(G, \operatorname{Aut}(G)) = \frac{2}{p}$ if and only if*

$$G = \begin{cases} \mathbb{Z}_p, & \text{if } p \text{ is any prime} \\ \mathbb{Z}_2 \times \mathbb{Z}_p, & \text{if } p \text{ is an odd prime.} \end{cases}$$

Theorem 2.3. *[AK17, Proposition 7] Let p be a prime and G be a group of order p^2. Then*

$$\Pr(G, \operatorname{Aut}(G)) = \frac{k}{p^2},$$

where k is either 2 or 3.

Theorem 2.4. *[AK17, Proposition 8] Let p be an odd prime and G be a group of order p^3. Then*

$$\Pr(G, \operatorname{Aut}(G)) = \frac{k}{p^3},$$

where $k \in \{2, 3, 4, p + 2\}$.

Theorem 2.5. *[AK17, Proposition 9] Let p be an odd prime and G be a abelian group of order p^4. Then*

$$\Pr(G, \operatorname{Aut}(G)) = \frac{k}{p^4},$$

where $k \in \{2, 3, 4, 5, 6\}$.

Let $[G, \mathrm{Aut}(G)] = \langle [x, \alpha] : x \in G \text{ and } \alpha \in \mathrm{Aut}(G) \rangle$ and $L(G) = \{x : [x, \alpha] = 1 \text{ for all } \alpha \in \mathrm{Aut}(G)\}$ be the auto-commutator subgroup and absolute center of G respectively. Then we have the following bounds for $\Pr(G, \mathrm{Aut}(G))$ in terms of $[G, \mathrm{Aut}(G)]$ and $L(G)$

Theorem 2.6. *[RS14, Lemma 2.1] Let G be a finite group. If p is the smallest prime dividing $|G|$ then*

$$\frac{1}{|[G, \mathrm{Aut}(G)]|}\left(\frac{|[G, \mathrm{Aut}(G)]| - 1}{|G : L(G)|} + 1\right) \leq \Pr(G, \mathrm{Aut}(G)) \leq \frac{p-1}{p|\mathrm{Aut}(G)|} + \frac{1}{p}.$$

A generalization of autocommuting probability of a finite group

The autocommutator of $x \in G$ and $\alpha \in \mathrm{Aut}(G)$ is defined as $[x, \alpha] := x^{-1}\alpha(x)$. Motivated by [PS08], Dutta and Nath [DN18] initiated the study of g-autocommuting probability of a finite group G. g-autocommuting probability of G is the probability that the autocommutator of a randomly chosen pair of elements, one from G and the other from $\mathrm{Aut}(G)$, is equal to a given element $g \in G$. That is

$$\Pr_g(G, \mathrm{Aut}(G)) = \frac{|\{(x, \alpha) \in G \times \mathrm{Aut}(G) : [x, \alpha] = g\}|}{|G||\mathrm{Aut}(G)|}.$$

Clearly, $\Pr_1(G, \mathrm{Aut}(G)) = \Pr(G, \mathrm{Aut}(G))$. Hence $\Pr(G, \mathrm{Aut}(G))$ generalizes $\Pr_g(G, \mathrm{Aut}(G))$. Dutta and Nath obtained the following computing formulae for $\Pr_g(G, \mathrm{Aut}(G))$.

Theorem 2.7. *[DN18, Theorem 2.3] Let G be a finite group. If $g \in G$ then*

$$\Pr_g(G, \mathrm{Aut}(G)) = \frac{1}{|G||\mathrm{Aut}(G)|}\sum_{\substack{x \in G \\ xg \in \mathrm{orb}(x)}} |C_{\mathrm{Aut}(G)}(x)| = \frac{1}{|G|}\sum_{\substack{x \in G \\ xg \in \mathrm{orb}(x)}} \frac{1}{|\mathrm{orb}(x)|}.$$

Theorem 2.8. *[DN18, Theorem 2.3] Let G and H be two finite groups such that $\gcd(|G|, |H|) = 1$. If $(g, h) \in G \times H$ then*

$$\Pr_{(g,h)}(G \times H, \mathrm{Aut}(G \times H)) = \Pr_g(G, \mathrm{Aut}(G))\Pr_h(H, \mathrm{Aut}(H)).$$

Dutta and Nath also found some bounds for the ratio $\Pr_g(G, \mathrm{Aut}(G))$. Some of their bounds are as follows.

Theorem 2.9. *[DN18, Proposition 3.1] Let G be a finite group. Then*

1. *If $g = 1$ then $\Pr_g(G, \mathrm{Aut}(G)) \geq \frac{|L(G)|}{|G|} + \frac{|C_{\mathrm{Aut}(G)}(G)|(|G| - |L(G)|)}{|G||\mathrm{Aut}(G)|}$.*
2. *If $g \neq 1$ then $\Pr_g(G, \mathrm{Aut}(G)) \geq \frac{|L(G)||C_{\mathrm{Aut}(G)}(G)|}{|G||\mathrm{Aut}(G)|}$.*

Theorem 2.10. *[DN18, Proposition 3.2] Let G be a finite group. Then*

$$\Pr_g(G, \mathrm{Aut}(G)) \leq \Pr(G, \mathrm{Aut}(G)).$$

The equality holds if and only if $g = 1$.

Theorem 2.11. *[DN18, Proposition 3.3] Let G be a finite group and p the smallest prime dividing $|\mathrm{Aut}(G)|$. If $g \neq 1$ then*

$$\Pr_g(G, \mathrm{Aut}(G)) \leq \frac{|G| - |L(G)|}{p|G|} < \frac{1}{p}.$$

Moghaddam et al. [ME] have defined autoisoclinism between two groups. Two groups G and H are said to be autoisoclinic if there exist isomorphisms $\psi : \frac{G}{L(G)} \to \frac{H}{L(H)}$, $\beta : K(G) \to K(H)$ and $\gamma : \mathrm{Aut}(G) \to \mathrm{Aut}(H)$ such that the following diagram commutes

$$\begin{array}{ccc} \frac{G}{L(G)} \times \mathrm{Aut}(G) & \xrightarrow{\psi\times\gamma} & \frac{H}{L(H)} \times \mathrm{Aut}(H) \\ \Big\downarrow{\scriptstyle a_{(G,\mathrm{Aut}(G))}} & & \Big\downarrow{\scriptstyle a_{(H,\mathrm{Aut}(H))}} \\ K(G) & \xrightarrow{\beta} & K(H) \end{array}$$

where $K(G)$ is the set of all autocommutaors of G and the maps $a_{(G,\mathrm{Aut}(G))} : \frac{G}{L(G)} \times \mathrm{Aut}(G) \to K(G)$ and $a_{(H,\mathrm{Aut}(H))} : \frac{H}{L(H)} \times \mathrm{Aut}(H) \to K(H)$ are given by

$$a_{(G,\mathrm{Aut}(G))}(xL(G), \alpha_1) = [x, \alpha_1] \text{ and } a_{(H,\mathrm{Aut}(H))}(yL(H), \alpha_2) = [y, \alpha_2]$$

respectively. In this case, the pair $(\psi \times \gamma, \beta)$ is called an autoisoclinism between the groups G and H. Dutta and Nath prove the following.

Theorem 2.12. *[DN18, Proposition 5.1] Let G and H be two finite groups and $(\psi \times \gamma, \beta)$ an autoisoclinism between them. Then*

$$\mathrm{Pr}_g(G, \mathrm{Aut}(G)) = \mathrm{Pr}_{\beta(g)}(H, \mathrm{Aut}(H)).$$

3 Relative autocommuting probability of a finite group

Moghaddam et al. [MK11] generalized $\mathrm{Pr}(G, \mathrm{Aut}(G))$ considering a subgroup H of G and defined relative autocommuting probability of a subgroup H of G denoted by $\mathrm{Pr}(H, \mathrm{Aut}(G))$. $\mathrm{Pr}(H, \mathrm{Aut}(G))$ is defined as

$$\mathrm{Pr}(H, \mathrm{Aut}(G)) = \frac{|\{(x, \alpha) \in H \times \mathrm{Aut}(G) : \alpha(x) = x\}|}{|H||\mathrm{Aut}(G)|}.$$

They obtained the following bounds.

Theorem 3.1. *[MK11, Theorem 2.3] Let H be a subgroup of a finite group G and p be the smallest prime number dividing $|\mathrm{Aut}(G)|$. Then*

$$\frac{|L(G) \cap H|}{|H|} + \frac{p(|H| - |L(G) \cap H|)}{|H||\mathrm{Aut}(G)|} \le \mathrm{Pr}(H, \mathrm{Aut}(G)) \le \frac{1}{p} + \frac{p-1}{p}\frac{|L(G) \cap H|}{|H|}.$$

Let $C_G(\alpha) = \{x \in G : [x, \alpha] = 1\}$. Rismanchian and Sepehrizadeh in [RS14] proved the following result.

Theorem 3.2. *[RS14, Theorem 2.3] Let H and K be two subgroups of a finite group G such that $H \subseteq K$. Then*

$$\mathrm{Pr}(K, \mathrm{Aut}(G)) \le \mathrm{Pr}(H, \mathrm{Aut}(G)).$$

The equality holds if and only if $K = HC_K(\alpha)$ for all $\alpha \in \mathrm{Aut}(G)$.

The following characterization for H in terms of $\mathrm{Pr}(H, \mathrm{Aut}(G))$ is obtained by Moghaddam et al. [MK11].

Theorem 3.3. *[MK11, Theorem 2.4] Let H be a subgroup of a finite group G and* $\Pr(H, \operatorname{Aut}(G)) = \frac{3}{4}$. *Then*

$$\frac{H}{L(G) \cap H} \cong \mathbb{Z}_2.$$

A generalization of relative autocommuting probability of a finite group

Dutta and Nath further generalized the notion of $\Pr_g(G, \operatorname{Aut}(G))$ in [DN]. They define

$$\Pr_g(H, \operatorname{Aut}(K)) := \frac{|\{(x, \alpha) \in H \times \operatorname{Aut}(K) : [x, \alpha] = g\}|}{|H||\operatorname{Aut}(K)|}$$

where $g \in K$. That is, $\Pr_g(H, \operatorname{Aut}(K))$ is the probability that the autocommutator of a randomly chosen pair of elements, one from H and the other from $\operatorname{Aut}(K)$, is equal to a given element $g \in K$. The ratio $\Pr_g(H, \operatorname{Aut}(K))$ is called generalized g-autocommuting probability of G relative to its subgroups H and K. They have obtained the following computing formula for $\Pr_g(H, \operatorname{Aut}(K))$.

Theorem 3.4. *[DN, Theorem 3.2] Let H and K be two subgroups of a finite group G such that $H \subseteq K$. If $g \in K$ then*

$$\begin{aligned}\Pr_g(H, \operatorname{Aut}(K)) =& \frac{1}{|H||\operatorname{Aut}(K)|} \sum_{\substack{x \in H \\ xg \in \operatorname{orb}_K(x)}} |C_{\operatorname{Aut}(K)}(x)| \\ =& \frac{1}{|H|} \sum_{\substack{x \in H \\ xg \in \operatorname{orb}_K(x)}} \frac{1}{|\operatorname{orb}_K(x)|}.\end{aligned}$$

They also obtained some bounds for the ratio considering $g = 1$. Some of the bounds are as follows.

Theorem 3.5. *[DN, Theorem 3.10] Let H and K be two subgroups of a finite group G such that $H \subseteq K$ and p the smallest prime dividing $|\operatorname{Aut}(K)|$. Then*

$$\Pr(H, \operatorname{Aut}(K)) \geq \frac{|L(H, \operatorname{Aut}(K))|}{|H|} + \frac{p(|H| - |X_H| - |L(H, \operatorname{Aut}(K))|) + |X_H|}{|H||\operatorname{Aut}(K)|}$$

and

$$\Pr(H, \operatorname{Aut}(K)) \leq \frac{(p-1)|L(H, \operatorname{Aut}(K))| + |H|}{p|H|} - \frac{|X_H|(|\operatorname{Aut}(K)| - p)}{p|H||\operatorname{Aut}(K)|},$$

where $L(H, \operatorname{Aut}(K)) = \{x \in H : [x, \alpha] = 1$ for all $\alpha \in \operatorname{Aut}(K)\}$ and $X_H = \{x \in H : C_{\operatorname{Aut}(K)}(x) = \{I\}\}$.

Theorem 3.6. *[DN, Theorem 3.14] Let H and K be two subgroups of a finite group G such that $H \subseteq K$. Then*

$$\Pr(H, \operatorname{Aut}(K)) \geq \frac{1}{|S(H, \operatorname{Aut}(K))|}\left(1 + \frac{|S(H, \operatorname{Aut}(K))| - 1}{|H : L(H, \operatorname{Aut}(K))|}\right)$$

where $S(H, \operatorname{Aut}(K)) = \{[x, \alpha] : x \in H$ and $\alpha \in \operatorname{Aut}(K)\}$. The equality holds if and only if $\operatorname{orb}_K(x) = xS(H, \operatorname{Aut}(K))$ for all $x \in H \setminus L(H, \operatorname{Aut}(K))$.

Theorem 3.7. *[DN, Corollary 3.15] Let H and K be two subgroups of a finite group G such that $H \subseteq K$. Then*
$$\Pr(H, \mathrm{Aut}(K)) \geq \frac{1}{|[H, \mathrm{Aut}(K)]|}\left(1 + \frac{|[H, \mathrm{Aut}(K)]| - 1}{|H : L(H, \mathrm{Aut}(K))|}\right)$$
where $S(H, \mathrm{Aut}(K)) = \{[x, \alpha] : x \in H \text{ and } \alpha \in \mathrm{Aut}(K)\}$. If $H \neq L(H, \mathrm{Aut}(K))$ then the equality holds if and only if $[H, \mathrm{Aut}(K)] = S(H, \mathrm{Aut}(K))$ and $\mathrm{orb}_K(x) = x[H, \mathrm{Aut}(K)]$ for all $x \in H \setminus L(H, \mathrm{Aut}(K))$.

Using these bounds, they further characterize H in terms of $\Pr(H, \mathrm{Aut}(K))$.

Theorem 3.8. *[DN, Theorem 3.11] Let $H \subseteq K$ be two subgroups of a finite group G.*

1. *If p and q are the smallest primes dividing $|\mathrm{Aut}(K)|$ and $|H|$ respectively then*
$$\Pr(H, \mathrm{Aut}(K)) \leq \frac{p + q - 1}{pq}.$$
In particular, if $p = q$ then
$$\Pr(H, \mathrm{Aut}(K)) \leq \frac{2p - 1}{p^2} \leq \frac{3}{4}.$$
2. *If $\Pr(H, \mathrm{Aut}(K)) = \frac{p+q-1}{pq}$, for some primes p and q, then pq divides $|H||\mathrm{Aut}(K)|$. Further, if p and q are the smallest primes dividing $|\mathrm{Aut}(K)|$ and $|H|$ respectively, then*
$$\frac{H}{L(H, \mathrm{Aut}(K))} \cong \mathbb{Z}_q.$$
In particular, if H and $\mathrm{Aut}(K)$ are of even order and $\Pr(H, \mathrm{Aut}(K)) = \frac{3}{4}$ then
$$\frac{H}{L(H, \mathrm{Aut}(K))} \cong \mathbb{Z}_2.$$

Theorem 3.9. *[DN, Theorem 3.12] Let $H \subseteq K$ be two subgroups of a finite group G.*

1. *If p, q are the smallest primes dividing $|\mathrm{Aut}(K)|$ and $|H|$ respectively and H is non-abelian then*
$$\Pr(H, \mathrm{Aut}(K)) \leq \frac{q^2 + p - 1}{pq^2}.$$
In particular, if $p = q$ then
$$\Pr(H, \mathrm{Aut}(K)) \leq \frac{p^2 + p - 1}{p^3} \leq \frac{5}{8}.$$
2. *If H is non-abelian and $\Pr(H, \mathrm{Aut}(K)) = \frac{q^2+p-1}{pq^2}$, for some primes p and q, then pq divides $|H||\mathrm{Aut}(K)|$. Further, if p and q are the smallest primes dividing $|\mathrm{Aut}(K)|$ and $|H|$ respectively then*
$$\frac{H}{L(H, \mathrm{Aut}(K))} \cong \mathbb{Z}_q \times \mathbb{Z}_q.$$
In particular, if H and $\mathrm{Aut}(K)$ are of even order and $\Pr(H, \mathrm{Aut}(K)) = \frac{5}{8}$ then
$$\frac{H}{L(H, \mathrm{Aut}(K))} \cong \mathbb{Z}_2 \times \mathbb{Z}_2.$$

4 Bibliography

[AK17] H. Arora and R. Karan. What is the probability an automorphism fixes a group element? *Comm. Algebra*, 45(3):1141–1150, 2017.

[DN] P. Dutta and R. K. Nath. Generalized autocommuting probability of a finite group relative to its subgroups. *Hacet. J. Math. Stat.*

[DN10] A. K. Das and R. K. Nath. On generalized relative commutativity degree of a finite group. *Int. Electron. J. Algebra*, 7(2):140–151, 2010.

[DN18] P. Dutta and R. K. Nath. Autocommuting probabilty of a finite group. *Comm. Algebra*, 46(3):961–969, 2018.

[DP13] Nath R. K. Das, A. K. and M. R. Pournaki. A survey on the estimation of commutativity in finite groups. *Southeast Asian Bull. Math.*, 37(2):161–180, 2013.

[EL07] Rezaei R. Erfanian, A. and P. Lescot. On the relative commutativity of a subgroup of a finite group. *Comm. Algebra*, 5, 2007.

[ETa68] P. Erdös and P. Tur´an. On some problems of a statistical group-theory. iv. *Acta. Math. Acad. Sci. Hungar.*, 19, 1968.

[ME] Sadeghifard M. J. Moghaddam, M. R. R. and M. Eshrati. Some properties of autoisoclinism of groups.

[MK11] Saeedi F. Moghaddam, M. R. R. and E. Khamseh. The probability of an automorphism fixing a subgroup element of a finite group. *Asian-Eur. J. Math.*, 4(2):301–308, 2011.

[ND10] R. K. Nath and A. K. Das. On a lower bound of commutativity degree. *Rend. Circ. Mat. Palermo*, 59(1):137–142, 2010.

[NY15] R. K. Nath and M. K. Yadav. Some results on relative commutativity degree. *Rend. Circ. Mat. Palermo*, 64(2):229–239, 2015.

[PS08] M. R. Pournaki and R. Sobhani. Probability that the commutator of two group elements is equal to a given element. *J. Pure Appl. Algebra*, 212, 2008.

[RL14] Erfanian A. Rezaei, R. and P. Lescot. A note on the relative commutativity degree of finite groups. *Asian-Eur. J. Math.*, 7(1):1450017–1–1450017–9, 2014.

[RS14] M. R. Rismanchian and Z. Sepehrizadeh. Autoisoclinism classes and autocommutativity degrees of finite groups. *Hacet. J. Math. Stat.*, 4(4):893–899, 2014.

[She75] G. J. Sherman. What is the probability an automorphism fixes a group element? *Amer. Math. Monthly*, 82, 1975.

A survey on clean rings

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Abstract. In this chapter, we shall highlight the outcomes of a survey on clean rings. Results stated here can be used to develop some further ideals and will help to solve the open questions mentioned in this chapter. We mainly focus on clean rings, its subclass and generalization.

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1 INTRODUCTION

Throughout our discussion, unless or otherwise explicitly stated, R will denote an associative ring with unity. We will use the symbols $\text{vnr}(R), \text{Nil}(R), \text{U}(R)$ and $\text{Idem}(R)$ respectively to denote the set of all von Neumann regular elements, nilpotent elements, units and idempotents of R. Also $J(R)$ will denote the Jacobson radical of R. Let M be a left R module. We denote the endomorphism ring of M by $end(M)$ and denote the ring of $n \times n$ matrices over the ring R by $M_n(R)$. A ring R is a $\star$-ring (or ring with involution) if there exists an operation $\star : R \to R$ such that for all $x, y \in R$, $(x+y)^\star = x^\star + y^\star$, $(xy)^\star = y^\star x^\star$ and $(x^\star)^\star = x$. An element p of a $\star$-ring is a projection if $p^2 = p = p^\star$. Obviously, 0 and 1 are projections of any $\star$-ring. Henceforth $P(R)$ will denote the set of all projections in a $\star$-ring. A ring R is said to have stable range one if for any $a, b \in R$ with $aR + bR = R$ there exists $y \in R$ such that $a + by \in U(R)$. In 1936, von Neumann defined that an element $a \in R$ is regular or von Neumann regular if $a = aba$ for some $b \in R$. Similarly an element $a \in R$ is called unit regular if $a = aua$ for some $u \in \text{U}(R)$ or equivalently $a = eu$ for some $e \in \text{Idem}(R)$ and $u \in \text{U}(R)$. In 1939, McCoy [McC39], generalized von Neumann regular rings to π-regular rings. A ring is said to be π-regular, if for each element $a \in R$, some positive integral power of a is von Neumann regular. That is, for each $a \in R$ there exist an element $x \in R$ and a positive integer n such that $a^n = a^n x a^n$. An element $a \in R$ is said to be strongly π-regular if there exist an element x and a positive integer n such that $a^n = a^{n+1}x$ (Azumaya [Azu54], called such an element right π-regular and similarly he defined left π-regular, further he called an element strongly π-regular if it is both left and right π-regular). Later on F. Dischinger showed that this definition is right and left symmetric. A ring R is strongly π-regular if all its elements are so. In 1977, W.K. Nicholson [Nic77], introduced the notion of clean element, which is an element that can be expressed as a sum of an idempotent and a unit. Clearly, the concept of clean element is an additive analogue of unit regular element. A ring is called a clean ring if each of its element is clean. A ring R is called strongly clean if every $a \in R$ can be expressed as $a = e + u$, where $e \in \mathrm{Idem}(R)$ and $u \in \mathrm{U}(R)$ with $eu = ue$. A.J. Diesl [Die06], in his Ph.D. thesis introduced the concept of nil clean rings, which is a ring in which every element of R can be expressed as a sum of an idempotent and a nilpotent element of R. It is easy to see that a nil clean ring is always a clean ring and of course the converse is not true. In 2006, M.S. Ahn and D.D. Anderson [AA06], defined the concept of weakly clean ring and almost clean ring. They defined that a ring R is weakly clean ring if for every $x \in R$, either $x = e + u$ or $x = -e + u$, where $e \in \mathrm{Idem}(R)$ and $u \in \mathrm{U}(R)$. Also they defined that a ring R to be almost clean ring if each $x \in R$ can be written as $x = r + e$, where $r \in \mathrm{vnr}(R)$ and $e \in \mathrm{Idem}(R)$. In 2002, H. Chen and M. Chen [CC03], defined the notion of clean ideals of a ring. An ideal I of a ring R is called clean ideal if every element of I is clean element of R. We mention some open questions given below:

1. Whether every nil clean element of a ring is clean?
2. If R is strongly nil clean and $e \in R$ is an idempotent, is the Peirce corner eRe strongly nil clean?
3. Does every strongly nil clean ring have stable range one?

2 Some works related to clean ring and its subclass

In this section, we mention some results related to clean rings and its subclass. Nil clean rings forms a subclass of clean rings and clean rings forms a subclass of almost clean rings. Here we mainly survey on nil clean rings, clean rings and almost clean rings.

- In 1977, W.K. Nicholson [Nic77] defined the concept of clean ring as a subclass of exchange ring. An element x of a ring R is said to be clean element if $x = u + e$, where $u \in U(R)$ and $e \in Idem(R)$. Ring R is said to be clean ring if all the elements of R are clean element. W. K. Nicholson also defined the notion of suitable ring. A ring R is called suitable if for each $x \in R$, there exists $e \in \mathrm{Idem}(R)$ such that $e - x \in R(x - x^2)$. The connection between clean ring and suitable ring is given below:

  Proposition 2.1. *Every clean ring is suitable.*

  Converse is not true by Bergman's example [LYZ08].

  Proposition 2.2. *A ring with central idempotent is clean if and only if it is suitable.*

  Definition 2.3. *A ring R is said to be potent if idempotents can be lifted modulo $J(R)$ and every left ideal not contained in $J(R)$ contains a non zero idempotent.*

  Definition 2.4. *Idempotents can be lifted modulo a one sided ideal I of a ring R if, for any $x \in R$ with $x - x^2 \in I$, there exists an idempotent $e \in R$ such that $e - x \in I$.*

  Some results related to suitable rings are given below:

  Theorem 2.5. *If R is suitable and $e^2 = e \in R$ the ring eRe is suitable.*

Proposition 2.6. *A ring is suitable if and only if idempotents can be lifted modulo every left ideal.*

- A. J. Diesl [Die13] studied a new class of clean rings called nil clean ring in the year 2013. A ring R is called nil clean ring if for any $x \in R$, $x = n + e$, where $n \in \mathrm{Nil}(R)$ and $e \in \mathrm{Idem}(R)$, if $ne = en$ in the nil clean expression of x, then the ring is said to be strongly nil clean ring. He characterized strongly nil clean elements of $\mathrm{End}(M_R)$, where $\mathrm{End}(M_R)$ is the ring of endomorphisms of a R-module M, as follows:

Proposition 2.7. *Let R be a ring, and let M_R be a right $R-$module. An element $f \in \mathrm{End}(M_R)$ is strongly nil clean if and only if there exists a direct sum decomposition $M = A \oplus B$ such that A and B are f-invariant and such that $f|_A \in \mathrm{End}(A)$ is nilpotent and $(1-f)|_B \in \mathrm{End}(B)$ is nilpotent.*

A. J. Diesl also characterized a commutative nil clean ring in terms of boolean ring.

Theorem 2.8. *Let, R be a commutative ring. Then R is nil clean if and only if $R/J(R)$ is boolean and $J(R)$ is nil.*

Theorem 2.9. *Let R be a ring and I be an nilpotent ideal of R. Let $\bar{R} = R/I$. If a is an element of R such that $\bar{a}$ is strongly nil clean in $\bar{R}$, then a is strongly nil clean element in R.*

- H. Chen [Che11] characterized the strongly nil cleanness of 2×2 matrices over local rings. For commutative local rings, he characterized strongly nil cleanness in terms of solvability of quadratic equations. Some of their results are given below:

Theorem 2.10. *Let R be a local ring. Then $A \in M_2(R)$ is strongly nil clean if and only if A is nilpotent or $I_2 - A$ is nilpotent or A is similar to a matrix $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$, where $\lambda \in \mathrm{Nil}(R)$ and $\mu \in 1 + \mathrm{Nil}(R)$.*

Theorem 2.11. *Let R be a commutative local ring. Then the following are equivalent.*

(1) $A \in M_2(R)$ *is strongly nil clean.*

(2) *A is nilpotent or $I_2 - A$ is nilpotent or the equation $x^2 - tr(A).x + \det(A) = 0$ has a root in $\mathrm{Nil}(R)$ and a root in $1 + \mathrm{Nil}(R)$.*

Let R be a local ring and $T(R) = \{\begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}, \mid a_{11}, a_{21}, a_{22}, a_{23} \in R\}$, then $T(R)$ is a 3×3 subring of $M_3(R)$ under usual addition and multiplication. In fact, $T(R)$ possesses the similar form of both the ring of all lower triangular matrices and the ring of all upper triangular matrices.

Theorem 2.12. *Let R be a local ring. Then $A \in T(R)$ is strongly nil clean if and only if each $A_{ii} \in \mathrm{Nil}(R)$ or $1 + \mathrm{Nil}(R)$.*

- In 2013, H. Chen [Che13] characterized the strongly nil cleanness of matrices over projective-free rings in terms of the factorizations of their characteristic polynomials. Some of their results are the following:

Theorem 2.13. *Let $E = end(_R M)$ and let $\alpha \in E$, then the following statements are equivalent.*

(1) $\alpha \in E$ *is strongly nil clean;*

(2) There exists $\pi^2 = \pi \in E$ *such that* $\pi\alpha = \alpha\pi$, $\alpha\pi \in \mathrm{Nil}(\pi E \pi)$ *and* $(1-\alpha)(1-\pi) \in \mathrm{Nil}((1-\pi)E(1-\pi))$;

(3) $M = P \oplus Q$, *where* P *and* Q *are* α*-invariant and* $\alpha|_P \in \mathrm{Nil}(end(P))$ *and* $(1-\alpha)|_Q \in \mathrm{Nil}(end(Q))$.

(4) $M = P_1 \oplus P_2 \oplus \cdots \oplus P_n$ *for some* $n \geq 1$, *where* P_i *is* α*-invariant and* $\alpha|_{P_i}$ *is strongly nil clean in* $end(P_i)$, *for each* i.

Definition 2.14. *For* $r \in R$, *define*

$$\mathbb{P}_r = \{f \in R[t] \mid f\, is\, monic,\, and\, f - (t-r)^{\deg(f)} \in \mathrm{Nil}(R)[t]\}$$

Theorem 2.15. *Let* R *be a commutative ring, let* $\phi \in M_n(R)$ *and let* $h \in R[t]$ *be a monic polynomial of degree* n. *If* $h(\phi) = 0$ *and there exists a factorization* $h = h_0 h_1$ *such that* $h_0 \in \mathbb{P}_0$ *and* $h_1 \in \mathbb{P}_1$, *then* ϕ *is strongly nil clean.*

An R module M is called free if it has a basis. Also an R module P is a projective module if there exists an R module Q such that $P \oplus Q$ is a free R module.

Theorem 2.16. *Let* R *be a projective-free ring, and let* $h \in R[t]$ *be a monic polynomial of degree* n. *Then the following are equivalent:*

(1) Every $\phi \in M_n(R)$ *with* $\chi(\phi) = h$ *is strongly nil clean.*

(2) There exists $\phi \in M_n(R)$ *with* $\chi(\phi) = h$ *is strongly nil clean.*

(3) There exists a factorization $h = h_0 h_1$ *such that* $h_0 \in \mathbb{P}_0$ *and* $h_1 \in \mathbb{P}_1$.

As a corollary, we have the following result.

Corollary 2.17. *Let* R *be a projective-free ring and let* $\phi \in M_n(R)$. *Then the following are equivalent:*

(1) ϕ *is strongly nil clean.*

(2) ϕ *is similar to* $\begin{pmatrix} \phi_0 & \\ & \phi_1 \end{pmatrix}$, *where* $\phi_0 \in M_r(R)$ *and* $I_{n-r} - \phi_1 \in M_{n-r}(R)\,(0 \leq r \leq n)$ *are nilpotent.*

Example 2.18. *Let* $\mathbb{Z}_4 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}$ *and let* $A = \begin{pmatrix} \bar{3} & \bar{3} & \bar{0} & \bar{0} \\ \bar{0} & \bar{0} & \bar{0} & \bar{0} \\ \bar{0} & \bar{0} & \bar{3} & \bar{3} \\ \bar{0} & \bar{0} & \bar{0} & \bar{3} \end{pmatrix} \in M_4(\mathbb{Z}_4)$. *Obviously* $\mathbb{Z}_4$ *is commutative local ring with* $\mathrm{Nil}(\mathbb{Z}_4) = \{\bar{0}, \bar{2}\}$ *is nil. Hence* $\mathbb{Z}_4$ *is projective-free.*

- In the year 2006, M.S. Ahn and D.D. Anderson [AO08] introduced weakly clean rings and almost clean rings. A ring R is weakly clean for $a \in R$, there exist some $e \in \text{Idem}(R)$ and $u \in \text{U}(R)$, such that if $a = u + e$ or $u - e$. For any subset S of a ring R, R is said to be S-weakly clean ring if for any $x \in R$, $x = u + e$ or $x = u - e$, where $u \in U(R)$ and $e \in S$. They proved that if R is weakly clean but not clean and $\text{Idem}(R) = \{0, 1\}$ then R has exactly two maximal ideals and $2 \in U(R)$. A ring R is called almost clean ring if for any $x \in R$, $x = r + e$, where $r \in \text{vnr}(R)$ and $e \in \text{Idem}(R)$. Clearly every clean ring is almost clean but an integral domain, which is always almost clean, is clean if and only if it is quasilocal. They also determined the indecomposable almost clean rings. Some results of weakly clean ring and almost clean ring are given bellow:

  Lemma 2.19. *(1) If R is weakly clean or $\{0, 1\}$-weakly clean, then so is every homomorphic image of R.*
  (2) If R is $\{0, 1\}$-weakly clean, then R has at most two maximal ideals.
  (3) Let K_1 and K_2 be fields. Then $K_1 \times K_2$ is $\{0, 1\}$-weakly clean if and only if both K_1 and K_2 have characteristics not equal to 2.

  Theorem 2.20. *A ring R is $\{0, 1\}$-weakly clean if and only if either*
  (1) R is quasilocal, or
  (2) R has exactly two maximal ideals and $2 \in U(R)$.

  A direct product $\prod R_\alpha$ of rings is clean if and only if each R_α is clean. M.S. Ahn and D.D. Anderson [AO08], determined when $\prod R_\alpha$ is weakly clean.

  Theorem 2.21. *Let $\{R_\alpha\}$ be a collection of commutative rings. Then the direct product $R = \prod R_\alpha$ is weakly clean if and only if each R_α is weakly clean ring and at most one R_α is not clean.*

  Theorem 2.22. *Suppose that the commutative ring R is a finite direct product of indecomposable rings, e.g., R is Noetherian. Then the following conditions are equivalent:*
  (1) R is almost clean.
  (2) For prime ideals $P, Q \subset Z(R)$ with $P + Q = R$, then there exists an idempotent e with $e \in P$ and $1 - e \in Q$.

  Theorem 2.23. *A commutative ring R satisfies $R = \text{vnr}(R) \cup \text{Idem}(R) \cup -\text{Idem}(R)$ if and only if R is isomorphic to one of the following:*
  (1) a domain,
  (2) a Boolean ring,
  (3) $\mathbb{Z}_3 \times B$, where B is a boolean ring, or
  (4) $\mathbb{Z}_3 \times \mathbb{Z}_3$.

- In [AN13], N. Ashrafi and E. Nasibi studied more about almost clean rings, they renamed almost clean ring by r-clean ring. They proved that the concepts of clean ring and almost clean ring are equivalent for abelian rings. So as a corollary we see that if 0 and 1 are the only idempotents in R, then an almost clean ring is an exchange ring.

Theorem 2.24. *Let R be an abelian ring. Then R is almost clean if and only if R is clean.*

Proposition 2.25. *Let R be an abelian ring and α be an endomorphism of R. Then the following are equivalent.*

1. *R is r-clean ring.*
2. *The formal power series ring $R[[x]]$ over R is r-clean.*
3. *The skew power series ring $R[[x;\alpha]]$ over R is r-clean.*

Theorem 2.26. *Let I be a regular ideal of a ring R and suppose that idempotents can be lifted modulo I. Then R is r-clean if and only if R/I is r-clean.*

Theorem 2.27. *Let A and B be rings, $M =_B M_A$, a bimodule and assume that one of the following holds:*

1. *A and B are clean.*
2. *one of the rings A and B is clean and the other one is r-clean.*

Then the formal triangular matrix ring $T = \begin{pmatrix} A & 0 \\ M & B \end{pmatrix}$ is r-clean.

- In 2003, H. Chen and M. Chen [CC03], introduced clean ideal of a ring which is a natural generalization of clean rings. An ideal I of a ring R is called clean ideal if every element of I is clean element. They proved that every matrix ideal over a clean ideal of a ring is clean. Also every ideal having stable range one of a regular ring is clean ideal.

Definition 2.28. *An ideal I of a ring R is said to be an exchange ideal if for any $x \in I$, there exists an idempotent $e \in I$ such that $e - x \in R(x - x^2)$.*

Some of their result about clean ideal is given below:

Theorem 2.29. *Let R be a unital ring and I an ideal in which every idempotent is central. Then the following are equivalent.*

(1) I is a clean ideal.

(2) I is an exchange ideal.

A finite orthogonal set of idempotents $e_1, e_2, \cdots, e_n$ in a ring R is said to be complete if $e_1 + e_2 + \cdots + e_n = 1 \in R$.

Proposition 2.30. *Let R be a unital ring and I an ideal of R. Then the following are equivalent:*

(1) I is clean ideal of R.

(2) there exists a complete set $\{e_1, e_2, \cdots, e_n\}$ of idempotents such that e_iIe_i is a clean ideal of e_iRe_i for all i.

3 Bibliography

[AA06] M. S. Ahn and D. D. Anddreson. Weakly clean rings and almost clean rings. *Rocky Mountain J. Maths.*, 36(11):783–798, 2006.

[AN13] N. Ashrafi and E. Nasibi. Rings in which elements are the sum of idempotent and a regular element. *Bull. Iranian Math. Soc.*, 39(3):579–588, 2013.

[AO08] O. Alkam and E. A. Osba. On the regular elements in $\mathbb{Z}_n$. *Turk. J. Math.*, 32:731–739, 2008.

[Azu54] G. Azumaya. Strongly π-regular rings. *Journal of the Faculty of Science, Hokkaido University. Ser. 1, Mathematics*, 13(1):034–039, 1954.

[CC03] H. Chen and M. Chen. On clean ideals. *Int. J. Math. Math. Sci.*, 62:3949–3956, 2003.

[Che11] H. Chen. Some strongly nil clean matrices over local rings. *Bull. Korean Math. Soc.*, 48:759–767, 2011.

[Che13] H. Chen. On strongly nil clean matrices. *Comm. Algebra*, 41:1074–1086, 2013.

[Die06] A. J. Diesl. *Classes of Strongly Clean Rings.* Ph.D thesis, University of California, 2006.

[Die13] A. J. Diesl. Nil clean rings. *Journal of Algebra*, 383:197–211, 2013.

[LYZ08] T.K. Lee, Z. Yi, and Y. Zhou. An example of bergman's and the extension problem for clean rings. *Comm. Algebra*, 36(4):1413–1418, 2008.

[McC39] N. H. McCoy. Generalized regular rings. *Bull. Amer. Math. Soc.*, 45(2):175–178, 1939.

[Nic77] W. K. Nicholson. Lifting idempotents and exchange rings. *Transactions of the American Mathematical Society*, 229:269–278, 1977.

Weak Nil Clean Rings

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Abstract. Peter V. Dancheva and W. Wm. McGovernb (see [3]) introduce the concept of a weak nil clean ring, a generalisation of nil clean ring, which is nothing but a ring with unity in which every element can be expressed as sum or difference of a nilpotent and an idempotent. Further if the idempotent and nilpotent commute the ring is called weak* nil clean. We characterize all $n \in \mathbb{N}$, for which $\mathbb{Z}_n$ is weak nil clean but not nil clean. Also we discuss S-weak nil clean rings and their properties, where S is a set of idempotents and show that if $S = \{0, 1\}$, then an S-weak nil clean ring contains a unique maximal ideal. Finally we show that weak* nil clean rings are exchange rings and strongly nil clean rings provided $2 \in R$ is nilpotent in the latter case.

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1 Introduction

Rings R are associative rings with unity unless otherwise indicated and modules are unitary. The Jacobson radical, group of units, set of idempotents and set of nilpotent elements of a ring R are denoted by $\mathrm{J}(R), \mathrm{U}(R), \mathrm{Idem}(R)$ and $\mathrm{Nil}(R)$ respectively. In the paper "Lifting idempotents and exchange rings"[4] Nicholson called an element r in a ring R clean element, if $r = e + u$ for some $e \in \mathrm{Idem}(R)$ and $u \in \mathrm{U}(R)$, and a ring is clean if every element of the ring is a clean element. Similarly a nil clean ring was introduced by Diesel [1] in his doctoral thesis and defined an element r in a ring R to be nil clean if $r = e + n$ for $e \in \mathrm{Idem}(R)$ and $n \in \mathrm{Nil}(R)$. A ring R is nil clean if each element of R is nil clean.

In the year 2006, Ahn and Andreson defined a ring R to be weakly clean if each element $r \in R$ can be written as $r = u + e$ or $r = u - e$ for some $u \in \mathrm{U}(R)$ and $e \in \mathrm{Idem}(R)$ [5]. Motivated by this concept, we observe the example $\mathbb{Z}_6 = \{\overline{0},\ \overline{1},\ \overline{2},\ \overline{3},\ \overline{4},\ \overline{5}\}$, here $\mathrm{Idem}(\mathbb{Z}_6) = \{\overline{0},\ \overline{1},\ \overline{3},\ \overline{4}\}$ and $\mathrm{Nil}(\mathbb{Z}_6) = \{\overline{0}\}$. So clearly $\mathbb{Z}_6$ is not a nil clean ring as $\overline{2}$ and $\overline{5}$ can not be written as sum of an idempotent and a nilpotent of $\mathbb{Z}_6$. But we see that every elements $r \in \mathbb{Z}_6$ can be written as $r = n - e$ or $r = n + e$ for $e \in \mathrm{Idem}(\mathbb{Z}_6)$ and $n \in \mathrm{Nil}(\mathbb{Z}_6)$, which led us to introduce weak nil clean ring. Weak nil clean ring a ring with unity in which each element of the ring can be expressed as sum or difference of a nilpotent and an idempotent.

A study on commutative weak nil clean rings have been done by Peter V. Dancheva and W. Wm. McGovernb (see [3]). Here we have given stronger version of few of its results along with some new results. We have also characterized all natural numbers n, for which $\mathbb{Z}_n$ is a weak nil clean ring but not nil clean ring. Further we have discussed S-weak nil clean ring, a ring in which each element can be expressed as sum or difference of a nilpotent and an element of S, where $S \subseteq \mathrm{Idem}(R)$ and have shown that if $S = \{0, 1\}$, then a S-weak nil clean ring contains a unique maximal ideal. Finally we have shown that weak* nil clean rings (Definition **??**) are exchange rings and strongly nil clean rings provided $2 \in R$ is nilpotent in the later case. We have ended the chapter by introducing weak J-clean rings and obtain few results on weak J-clean rings as an effort to answer **Problem** 5 of [3].

2 WEAK NIL CLEAN RING

Definition 2.1. *An element $r \in R$ is said to weak nil clean element of the ring R, if $r = n-e$ or $r = n - e$, for some $n \in \mathrm{Nil}(R)$ and $e \in \mathrm{Idem}(R)$ and a ring is said to be weak nil clean ring if each of its elements is weak nil clean. Further if $r = n - e$ or $n + e$ with $ne = en$, then r is called weak* nil clean.*

Obviously every nil clean ring is weak nil clean, but the above example denies the converse. Also if R is a weak nil clean ring or a weak* nil clean ring then for $n \geq 2$, $S = \{A \in T_n(R) : a_{11} = a_{22} = \cdots = a_{nn}\}$, is weak nil clean ring which is not weak* nil clean, where $T_n(R)$ is the ring of upper triangular matrices over R. Analogue to the concept of clean and nil clean rings, it is easy to see that every weak nil clean ring is weakly clean and the converse is not true.

Theorem 2.2. *Homomorphic image of a weak nil clean ring is weak nil clean.*

However the converse is not true as $\mathbb{Z}_6 \cong \mathbb{Z}/\langle 6 \rangle$ is a weak nil clean ring, but $\mathbb{Z}$ is not a weak nil clean ring. A finite direct product $\prod R_\alpha$ of rings is nil clean if and only if each R_α is nil clean. our next observe next result is on finite direct product of weak nil clean rings.

Theorem 2.3. *Let $\{R_\alpha\}$ be a finite collection of rings. Then the direct product $R = \prod R_\alpha$ is weak nil clean if and only if each R_α is weak nil clean and at most one R_α is not nil clean.*

Proof. ($\Rightarrow$) Let R be weak nil clean, then each R_α being homomorphic image of R is weak nil clean. Suppose for some α_1 and α_2, $\alpha_1 \neq \alpha_2$, R_{α_1} and R_{α_2} are not nil clean. Since R_{α_1} is not nil clean, not all elements $x \in R_{\alpha_1}$ are of the form $n - e$, where $n \in \mathrm{Nil}(R_{\alpha_1})$ and $e \in \mathrm{Idem}(R_{\alpha_1})$. But R_{α_1} is weak nil clean, so there exists $x_{\alpha_1} \in R_{\alpha_1}$, with $x_{\alpha_1} = n_{\alpha_1} + e_{\alpha_1}$, where $e_{\alpha_1} \in \mathrm{Idem}(R_{\alpha_1})$ and $n_{\alpha_1} \in \mathrm{Nil}(R_{\alpha_1})$, but $x_{\alpha_1} \neq n - e$ for any $n \in \mathrm{Nil}(R_{\alpha_1})$ and $e \in \mathrm{Idem}(R_{\alpha_1})$. Likewise there exists $x_{\alpha_2} \in R_{\alpha_2}$, with $x_{\alpha_2} = n_{\alpha_2} - e_{\alpha_2}$, where $e_{\alpha_2} \in \mathrm{Idem}(R_{\alpha_2})$ and $n_{\alpha_2} \in \mathrm{Nil}(R_{\alpha_2})$, but $x_{\alpha_2} \neq n + e$ for any $n \in \mathrm{Nil}(R_{\alpha_2})$ and $e \in \mathrm{Idem}(R_{\alpha_2})$.

$$\text{Define } x = (x_\alpha) \in R \text{ by } x_\alpha = \begin{cases} x_{\alpha_i} & \text{if } \alpha \in \{\alpha_1, \ \alpha_2\} \\ 0 & \text{if } \alpha \notin \{\alpha_1, \ \alpha_2\} \end{cases}$$

Then clearly $x \neq n \pm e$ for any $n \in \mathrm{Nil}(R)$ and $e \in \mathrm{Idem}(R)$, hence at most one R_α is not nil clean. ($\Leftarrow$) If each R_α is nil clean, then $R = \prod R_\alpha$ is nil clean, so weak nil clean. So assume some R_{α_0} is weak nil clean but not nil clean and that all other R_α's are nil clean. Let $x = (x_\alpha) \in R$. In R_{α_0}

we can write $x_{\alpha_0} = n_{\alpha_0} + e_{\alpha_0}$ or $x_{\alpha_0} = n_{\alpha_0} - e_{\alpha_0}$, where $n_{\alpha_0} \in \mathrm{Nil}(R_{\alpha_0})$, $e_{\alpha_0} \in \mathrm{Idem}(R_{\alpha_0})$. If $x_{\alpha_0} = n_{\alpha_0} + e_{\alpha_0}$, for $\alpha \neq \alpha_0$, let $x_\alpha = n_\alpha + e_\alpha$ and if $x_{\alpha_0} = n_{\alpha_0} - e_{\alpha_0}$, for $\alpha \neq \alpha_0$, let $x_\alpha = n_\alpha - e_\alpha$ then $n = (n_\alpha) \in \mathrm{Nil}(R)$ and $e = (e_\alpha) \in \mathrm{Idem}(R)$ and $x = n + e$ or $x = n - e$ respectively, hence R is weak nil clean. □

Proposition 2.4. *Let R be a weak nil clean ring, then $J(R) \subseteq \mathrm{Nil}(R)$.*

Proof. Let $x \in J(R)$. Then $x = n - e$ or $x = n + e$, where $n \in \mathrm{Nil}(R)$ and $e \in \mathrm{Idem}(R)$. If $x = n - e$ then there exists a $k \in \mathbb{N}$ such that $(x + e)^k = 0$, which gives $e \in J(R) \cap \mathrm{Idem}(R)$, hence $e = 0$ i.e., $x = n \in \mathrm{Nil}(R)$. Similarly for $x = n + e$, we get $x = n \in \mathrm{Nil}(R)$. Thus $J(R) \subseteq \mathrm{Nil}(R)$. □

Proposition 2.5. *Let R be a commutative ring. Then R is weak nil clean if and only if $R/J(R)$ is Boolean*

Proposition 2.6. *If a commutative ring R is weak nil clean, $R/\mathrm{Nil}(R)$ is weak nil clean and converse holds if idempotents can be lifted modulo* $\mathrm{Nil}(R)$.

Proof. ($\Rightarrow$) Follows from Theorem (2.2).
($\Leftarrow$) Let $x \in R$. Since $R/\mathrm{Nil}(R)$ is weak nil clean, so $x + \mathrm{Nil}(R) = y + \mathrm{Nil}(R)$ or $(-y) + \mathrm{Nil}(R)$, where $y^2 - y \in \mathrm{Nil}(R)$ (as $R/\mathrm{Nil}(R)$ is a reduced ring). Since idempotents of R lift modulo $\mathrm{Nil}(R)$, so there exist $e \in \mathrm{Idem}(R)$ such that $y - e \in \mathrm{Nil}(R)$, which implies $x - e \in \mathrm{Nil}(R)$ or $x + e \in \mathrm{Nil}(R)$ i.e., $x - e = n$ or $x + e = m$ for some $m, n \in \mathrm{Nil}(R)$, which proves the result. □

For more examples of weak nil clean rings, we consider the method of idealization. Let R be a commutative ring and M a left $R-$module. The idealization of R and M is the ring $R(M) = R \oplus M$ with product defined as $(r, m)(r', m') = (rr', rm' + r'm)$ and sum as $(r, m)(r', m') = (r + r', m + m')$, for (r, m), $(r', m') \in R(M)$.

Theorem 2.7. *Let R be a ring and M be a left R-module. Then R is weak nil clean if and only if $R(M)$ is weak nil clean.*

Proof. ($\Leftarrow$) Note that $R \approx R(M)/(0 \oplus M)$ is homomorphic image of $R(M)$. Hence by Theorem (2.2), R is weak nil clean ring.
($\Rightarrow$) Let R be weak nil clean ring and $(r, m) \in R \oplus M$, where $r \in R$ and $m \in M$, we have $r = n + e$ or $n - e$ for $n \in \mathrm{Nil}(R)$ and $e \in \mathrm{Idem}(R)$
$(r, m) = (n + e, m)$ or $(n - e, m) = (n, m) + (e, 0)$ or $(n, m) - (e, 0)$ is weak nil clean expression of (r, m), where $(n, m) \in \mathrm{Nil}(R)$ and $(e, 0) \in \mathrm{Idem}(R)$, hence $R(M) = R \oplus M$ is weak nil clean. □

Now we try to characterize all n for which $\mathbb{Z}_n$ is weak nil clean but not nil clean. We recall that, $\mathrm{Idem}\,(\mathbb{Z}_{p^k}) = \{0, 1\}$, for any prime $p \in \mathbb{N}$ and $k \in \mathbb{N}$.

Lemma 2.8. $\mathbb{Z}_{3^k}$, $k \in \mathbb{N}$ *is weak nil clean but not nil clean.*

Proof. The proof follows from the fact that $\mathrm{Idem}\,(\mathbb{Z}_{3^k}) = \{0, 1\}$ and $\mathrm{Nil}\,(\mathbb{Z}_{3^k}) = \{0, 3, 6, ..., 3(3^{k-1} - 1)\}$.

Lemma 2.9. $\mathbb{Z}_{p^k}, k \in \mathbb{N}$ *is weak nil clean but not nil clean, where p is prime iff $p = 3$.*

Proof. ($\Leftarrow$) It follows from Lemma 2.8
($\Rightarrow$) We know that $\mathbb{Z}_{2^k}$ is nil clean $\forall k \in \mathbb{N}$ and $\mathbb{Z}_{3^k}$ is weak nil clean $\forall k \in \mathbb{N}$ but not nil clean. Now consider $p > 3$, then we have $\mathrm{Idem}\,(\mathbb{Z}_{p^k}) = \{0, 1\}$ and $\mathrm{Nil}\,(\mathbb{Z}_{p^k}) = \{0, p, 2p, \ldots, (p^{k-1} - 1)p\}$. So if we consider the sum or difference of nilpotents and idempotents of $\mathbb{Z}_{p^k}$ respectively, then at most $4p^{k-1}$ elements can be obtained, but $p > 4$, so $p^k > 4p^{k-1}$. Hence all elements of $\mathbb{Z}_{p^k}$ can not be written as sum or difference of nilpotent and idempotent of $\mathbb{Z}_{p^k}$ respectively. So $p = 3$. $\square$

Theorem 2.10. *The only n for which $\mathbb{Z}_n$ is weak nil clean but not nil clean is of the form either 3^k or 2^r3^t, where $k, r, t \in \mathbb{N}$.*

Proof. We have already seen that $\mathbb{Z}_{3^k}$ is weak nil clean but not nil clean. Next let $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$ with $\alpha_i \in \mathbb{N},\ 1 \le i \le k$ and p_i's are distinct primes such that $p_1 \le p_2 \le \ldots \le p_n$. If $k > 2$, then there exists some i with $1 \le i \le k$ such that $p_i > 3$. Then $\mathbb{Z}_{p_i^{\alpha_i}}$ is not weak nil clean. Hence $\mathbb{Z}_n$ can not be weak nil clean as $\mathbb{Z}_n = \mathbb{Z}_{p_1^{\alpha_1}} \oplus \mathbb{Z}_{p_2^{\alpha_2}} \oplus \cdots \oplus \mathbb{Z}_{p_k^{\alpha_k}}$. So $k \le 2$ and $p_i \le 3$ i.e., $n = p_1^{\alpha_1}p_2^{\alpha_2}$. If $k = 1$, then p_1 must be 3 as $\mathbb{Z}_{2^k}$ is nil clean. Again if $k = 2$, then since p_i's are distinct so $p_1 = 2$ and $p_2 = 3$. Also if $n = 2^{\alpha_1}3^{\alpha_2}$, then $\mathbb{Z}_n = \mathbb{Z}_{2^{\alpha_1}} \oplus \mathbb{Z}_{3^{\alpha_2}}$. Since $\mathbb{Z}_{2^{\alpha_1}}$ is nil clean and $\mathbb{Z}_{3^{\alpha_2}}$ is weak nil clean but not nil clean, so $\mathbb{Z}_n$ is weak nil clean but not nil clean. This completes the proof.$\square$

The polynomial ring $R[x]$ over a weak nil clean ring is not necessarily weak nil clean. In fact if R is commutative the $R[x]$ is never weak nil clean. For then $x \in R[x]$ is of the form $\sum_i a_i x^i - e$ or $\sum_i a_i x^i + e$, where $a_i \in \mathrm{Nil}(R), e \in \mathrm{Idem}(R)$, giving $a_0 - e = 0$ or $a_0 + e = 0$, which is absurd.

However if R is weak nil clean and $\sigma : R \to R$ is a ring endomorphism then for any n, the quotient $S = R[x;\sigma]/ < x^n >$, where $R[x;\sigma]$ is the Hilbert twist, is a weak nil clean ring. Indeed if $f = a_0 + a_1x + a_2x^2 + \ldots + a_{n-1}x^{n-1} \in S$ and $a_0 = n + e$ or $a_0 = n - e$, where $n \in \mathrm{Nil}(R), e \in \mathrm{Idem}(R)$, then $f = (f - e) + e$ or $f = (f + e) - e$ is a weak nil clean decomposition of f in S.

In order to show that, weak* nil cleanness penetrate to corner, we need following lemmas.

Lemma 2.11. *Let R be a ring and $x = n + e$ or $n - e$ be a weak* decomposition of $x \in R$ with $n \in \mathrm{Nil}(R)$ and $e \in \mathrm{Idem}(R)$, then $\mathrm{ann}_l(a) \subseteq \mathrm{ann}_l(e)$ and $\mathrm{ann}_r(a) \subseteq \mathrm{ann}_r(e)$.*

Proof. Let $r \in \mathrm{ann}_l(x)$ then $rx = 0$. If $x = e + n$ then $rn + re = 0$, and so $rne + re = 0$ i.e., $re(n + 1) = 0$ implying $re = 0$ and thus $r \in \mathrm{ann}_l(e)$.
Again if $x = n - e$, then $rn - re = 0$ and so $rne - re = 0$ i.e., $re(n - 1) = 0$ implying $re = 0$ and thus $r \in \mathrm{ann}_l(e)$. Hence $\mathrm{ann}_l(a) \subseteq \mathrm{ann}_l(e)$. Similarly the other part i.e. $\mathrm{ann}_r(a) \subseteq \mathrm{ann}_r(e)$. $\square$

Lemma 2.12. *Let R be a ring and $x = n + e$ or $n - e$ be a weak* decomposition of $x \in R$ with $n \in \mathrm{Nil}(R)$ and $e \in \mathrm{Idem}(R)$, then $\mathrm{ann}_l(a) \subseteq R(1 - e)$ and $\mathrm{ann}_r(a) \subseteq (1 - e)R$.*

Proof. Let $r \in \mathrm{ann}_l(x)$ then by above Lemma 2.11 we have $r \in \mathrm{ann}_l(e)$ i.e. $re = 0$, So $r = r(1 - e) \in R(1 - e)$ and hence $\mathrm{ann}_l(a) \subseteq R(1 - e)$. Similarly we have $\mathrm{ann}_r(a) \subseteq (1 - e)R$. $\square$

Theorem 2.13. *Let R be a ring and $f \in Idem(R)$, then $x \in fRf$ is weak* nil clean in R if and only if x is weak* nil clean in fRf.*

Proof.($\Leftarrow$) If $x \in fRf$ is weak* nil clean in fRf, then by the same weak* nil clean decomposition, x is weak* nil clean in R.
($\Rightarrow$) Let x is weak* nil clean in R, so $x = n + e$ or $n - e$ for some $n \in \mathrm{Nil}(R)$ and $e \in \mathrm{Idem}(R)$ with

$ne = en$. First let $x = n + e$, since $x \in fRf$, so $(1-f) \in \mathrm{ann}_l(x) \cap \mathrm{ann}_r(x) \subseteq R(1-e) \cap (1-e)R = (1-e)R(1-e)$ [by Lemma 2.12]. So we have $(1-f)e = 0 = e(1-f)$ giving $fe = e = ef$, and consequently $fef \in \mathrm{Idem}(fRf)$. Also $xf = fx$, therefore we have $nf = fn$, i.e. $fnf \in \mathrm{Nil}(fRf)$. Hence $x = fnf + fef$. Similarly if $x = n - e$ then $x = fnf - fef$. Hence x is weak* nil clean in fRf.□

The following is an immediate consequence of this result.

Corollary 2.14. *Let R is weak* nil clean ring $e \in \mathrm{Idem}(R)$, then the corner ring eRe is also weak* nil clean.*

3 S-WEAK NIL CLEAN RING

S-weak nil clean ring is a generaliztion of weak nil clean rings which is defined as follows:

Definition 3.1. *let S be a nonempty set of idempotents of R, then R is called $S-weak\ \ nil\ \ clean$ if each $r \in R$ can be written as $r = n + e$ or $n - e$, where $n \in \mathrm{Nil}(R)$ and $e \in S$.*

Proposition 3.2. *Let R is $\{0,1\}$-weak nil clean ring, then R has exactly one maximal ideal.*

Proof. Let R be $\{0,1\}$-weak nil clean ring. Then $R = \mathrm{U}(R) \bigcup \mathrm{Nil}(R)$ and $\mathrm{U}(R) = (1 + \mathrm{Nil}(R)) \bigcup (-1 + \mathrm{Nil}(R))$. It follows that for any $x \in \mathrm{Nil}(R)$ and any $r \in R, xr, rx \in \mathrm{Nil}(R)$. Next if possible let $n_1 - n_2 = u$, where $n_1, n_2 \in \mathrm{Nil}(R)$ and $u \in \mathrm{U}(R)$. Then $u^{-1}n_1 - u^{-1}n_2 = 1$ i.e., $n_3 = 1 + n_4$, where $u^{-1}n_1 = n_3 \in \mathrm{Nil}(R)$ and $u^{-1}n_2 = n_4 \in \mathrm{Nil}(R)$, which is a contradiction as $n_3 \in \mathrm{Nil}(R)$. Thus $n_1 - n_2 \in \mathrm{Nil}(R)$, for any $n_1,\ \ n_2 \in \mathrm{Nil}(R)$ implying that $\mathrm{Nil}(R)$ is an ideal.□

From above theorem it is clear that $\{0,1\}-$ nil clean rings are local rings. Converse is not true.

Theorem 3.3. *If a ring R is S-weak* nil clean for $S \subseteq \mathrm{Idem}(R)$ then $S = \mathrm{Idem}(R)$.*

Proof. Let $e' \in \mathrm{Idem}(R)$, then $-e' \in R$. Since R is S-weak* nil clean, so $-e' = n+e$ or $-e' = n-e$ for some $n \in \mathrm{Nil}(R)$, and $e \in S$, with $ne = en$. If $-e' = n + e$, then $1 - e' = 1 + n + e$ i.e., $(1+n+e)^2 = 1+n+e$, which gives $1+n^2+e+2n+2e+2ne = 1+n+e$ i.e., $n^2+n+2e(1+n) = 0$, implies $(n+2e)(1+n) = 0$. But $1 + n \in U(R)$, so $n = -2e$, giving $-e' = n + e = -2e + e = -e$. Thus $e' = e \in S$.
Again if $-e' = n-e$, then $(-e')^2 = {e'}^2 = e'$ i.e., $(n-e)^2 = -n+e$, which gives $n^2-2ne+e = -n+e$ i.e., $n^2 + n(1-2e) = 0$, implies $n\{n + (1-2e)\} = 0$. But $n + (1-2e) \in U(R)$, so $n = 0$. So $e' = e \in S$. Hence $\mathrm{Idem}(R) = S$. □

But in case of weak clean ring it is possible that R is $S-$weak clean and $S \subsetneq \mathrm{Idem}(R)$.

4 MORE RESULTS ON WEAK NIL CLEAN RING

It is well known that $\mathbb{Z}_3$ is clean, so upper triangular matrix ring $\mathbb{T}_2(\mathbb{Z}_3)$ is clean and hence exchange, but $\mathbb{T}_2(\mathbb{Z}_3)$ is not weak nil clean ring, so in general, exchange ring are not weak nil clean ring. But one can see that weak* nil clean rings are exchange.

Theorem 4.1. *Let R be a weak* nil clean ring, then R is a exchange ring.*

Proof. Let R be a weak* nil clean ring and $x \in R$, then $x = n + e$ or $x = n - e$, where $n \in \mathrm{Nil}(R)$ and $e \in \mathrm{Idem}(R)$.
If $x = n - e$, then

$$\begin{aligned}(1-n)[x-(1-n)^{-1}e(1-n)] &= (1-n)[(n-e)-(1-n)^{-1}e(1-n)],\\ &= n-e-n^2+ne-e-en,\\ &= x-(n-e)^2 = x-x^2,\\ \text{implies} \qquad [x-(1-n)^{-1}e(1-n)] &= (1-n)^{-1}(x-x^2).\end{aligned}$$

Similarly if $x = n+e$, we have $x - e = u^{-1}(x^2 - x)$ for $u = (2e-1)+n \in \mathrm{U}(R)$. Then by condition (1) of Proposition 1.1 of [4], R is exchange. □

Finally we take the question " under what condition a weak* nil clean ring is strongly nil clean ring?" To answer this question we need following the Lemma.

Lemma 4.2. *Let R be a ring with and M_R be a right R-module. An endomorphism $\phi \in \mathrm{End}(M_R)$ is sum or difference of a nilpotent n and an idempotent e, which commutes with $2 \in Nil(R)$ then there exist a direct sum decomposition $M = A \oplus B$ such that $\phi|_A$ is an element of $\mathrm{End}(A)$ which is nilpotent and $(1-\phi)|_B$ is an element of $\mathrm{End}(B)$ which is nilpotent.*

Proof. Suppose $\phi = a - e$, where $e \in \mathrm{Idem}(\mathrm{End}(M_R))$ and $a \in \mathrm{Nil}(\mathrm{End}(M_R))$ and suppose $ea = ae$. We define decomposition $M = A \oplus B$, by setting
$A = (1-e)M$ and $B = eM$. Then A and B are $\phi-$invariant.
Now $\phi|_A = (a-e)|_A = a|_A - e|_A = a|_A$ and so $\phi|_A$ is nilpotent.
And $(1-\phi)|_B = (1-(a-e))|_B = (1-a+e)|_B = (2-a-(1-e))|_B = (2-a)|_B$ is nilpotent as 2 is nilpotent.
Again, if $\phi = a + e$, where $e \in \mathrm{Idem}(\mathrm{End}(M_R))$ and $a \in \mathrm{Nil}(\mathrm{End}(M_R))$, then by Definition 1.2.8 and Lemma 1.2.3 of [1] such a decomposition exists. □

Now we can state following theorem.

Theorem 4.3. *A ring R is strongly nil clean if and only if R is weak* nil clean with $2 \in \mathrm{Nil}(R)$.*

Proof.(⇒) Clear from the definition of weak* nil clean ring.
(⇐) The result follows from Lemma 1.2.6 of [1] and Lemma (4.2).

Corollary 4.4. *A weak* nil clean ring R with $2 \in \mathrm{Nil}(R)$, is strongly $\pi-$regular.*

5 Bibliography

[1] A. J. Diesl, *Classes of Strongly Clean Rings*, Ph D. Thesis, University of California, Berkeley, (2006).

[2] D. K. Basnet, J. Bhattacharyya, Weak Nil Clean Rings, arXiv:1510.07440, (2015).

[3] P. V Danchev, W Wm. McGovern. Commutative weakly nil clean unital rings, *Journal of Algebra.* (2015) *425: 410-422.*

[4] W.K. Nicholson, Lifting idempotents and exchange rings(^{1}). *Trans. Amer. Math. Soc*, (1977). $229 : 269 - 278$.

[5] M-S. Ahn, D.D. Anddreson, Weak clean rings and almost clean rings, *Rocky mount. Jorn. math.* 36, (2006) no 3, 783-798.

A study on primitive elements over finite fields

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Abstract. This chapter we will present some basic survey on some of the basic algebraic structures of groups, rings, fields (more specifically finite fields) and primitive elements. For finite field structures, we have to introduce some additional notations, symbols, and operations from elementary addition and multiplication operations. One of the most beautiful things in finite field theory is that the mystery behind its structure of generators, as till now there doesn't exist any certain process to determine the generators. Even if the exact structure can't be determined, one can determine the existence of such elements. So, in this chapter, we are going to discuss the methods of determining the existence of such generators of a finite field.

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1 Introduction

A law of composition is the concept of operation to arbitrary set S is to combine elements of the set. If $S \times S$ denote the set of all ordered pairs (a, b) with $a, b \in S$. Then a "binary operation" is a map as $S \times S \to S$ along with the condition that $(a, b) \in S \times S$ has an image in S, basically, this is the closure property for the operation.

In algebraic systems, a group is a system with a single associative operation that has been studied extensively and applied in other branches of pure as well as applied mathematics.

Definition 1.1. *A group is an ordered pair $(G, *)$, where G is a set $*$ is a binary operation with the following additional properties:*

(i) operation is associative i.e, $a * (b * c) = (a * b) * c$, where $a, b, c \in G$.

(ii) There is an identity element $'e'$ in G with respect to the binary operation as $a * e = e * a = a$, for all $a \in G$.

(iii) for each element $a \in G$, there exist an element a^{-1} such that $a * a^{-1} = a^{-1} * a = e$.

Along with the above properties, if the "commutative" property is added, then the group is called "Abelian" group.

Definition 1.2. *A group is called "cyclic" if the group can be generated by a single element, i.e, each element of the group can be expressed in terms of a particular element called "generator" of the group.*

Till now we are discussing algebraic structures with only one binary operation. But by considering two binary operations on the same set we can obtain a new algebraic structure with additional properties. In next stage we are focusing on such algebraic structures. Such structures are ring, integral domain,field etc., which have enormous applications in mathematics.

Definition 1.3. *A ring R is an algebraic structure with two binary operations $+$ and $\cdot$ namely addition and multiplication, denoted by $(R, +, \cdot)$, with following properties:*

(i) $(R, +)$ *is an abelian group.*

(ii) operation $\cdot$ *is associative, i.e,* $a \cdot (b \cdot c) = (a \cdot b) \cdot c$, *for all* $a, b, c \in R$.

(iii) distribution low holds on R, *i.e,* $a \cdot (b + c) = a \cdot b + a \cdot b$, *for all* $a, b, c \in R$.

In the above definition the operations "+" and "$\cdot$" not necessarily mean elementary addition and multiplication. Furthermore a ring is "ring with unity" if it has multiplicative identity. To distinguish between additive and multiplicative identity we use the notation "0" for additive and "1" for multiplicative identity respectively. Ring $(R, +, \cdot)$ is commutative if $\cdot$ is commutative over R. For example $(\mathbb{Z}, +, \cdot)$, $(\mathbb{Q}, +, \cdot)$, $(\mathbb{R}, +, \cdot)$, $(\mathbb{C}, +, \cdot)$ are examples of commutative rings with unity, where the symbols have usual meaning.

Definition 1.4. *An "integral domain" is a commutative ring with unity and without no non zero zero-divisors, i.e, if* $a \cdot b = 0$ *then either* $a = 0$ *or* $b = 0$. *Finally a commutative ring with unity which non zero elements also forms a group under* $\cdot$*(multiplication) operation is called a "Field".*

Fields can be categorised as two types, one is infinite fields and other is finite fields. In this chapter we are going to deal with finite fields only. Hence the following are based on finite fields.

A subfield K of a field F is a subset of F,which itself is a field with the same operations as F. In this context, F is called "extension field" of K. A filed is called "prime field" if it has no proper subfield. For finite fileds, a prime field must have prime order. Further "the characteristic" of a field F is the smallest positive integer n such that $n \cdot 1 = 0$. For finite field, one can easily determine that the characteristic must be prime. Based on this we have the following theorem,

Theorem 1.5. *A finite field F has p^n elements, where p is the characteristic of F and n is the degree of extension from its prime subfield.*

For every prime p and every positive integer n, there exist a finite field of order p^n. Any finite field $\mathbb{F}_q$ of order $q = p^n$ elements is isomorphic to the splitting field of $x^q - 1$ over $\mathbb{F}_p$. Furthermore, every subfield of $\mathbb{F}_q$ has order p^m , where m divides n.

With all this, we arrive at one of the most impotent theorems of this chapter, and our journey of finding generators (later we will describe them as primitive elements) begin fro here.

Theorem 1.6. *The multiplicative group $\mathbb{F}_q^*$ of a finite field $\mathbb{F}_q$ is cyclic.*

For further details about this theorem, readers are requested to see any book of finite field.

Since the multiplicative group is cyclic, the question of existence of generators arise and to solve this question the following are done.

2 Primitive Elements

We begin this section with definition of primitive element.

Definition 2.1. *A generator of the multiplicative group* $\mathbb{F}_q^*$ *is called a primitive element of* $\mathbb{F}_q$.

Now, here comes the most interesting part of this chapter. Any field of order q (i.e., $\mathbb{F}_q$) has $\phi(q-1)$ primitive elements, where ϕ is the Euler's phi-function. Even though there are $\phi(q-1)$ primitive elements in a finite field $\mathbb{F}_q$, but finding one such primitive element may be difficult, as there is no polynomial time algorithm to compute a primitive element, as mentioned earlier this is one of the most beautiful and mystical phenomena in finite field theory.

Since using the properties of primitive elements , modern day cryptosystems such as ElGamal crypto-system,*The Diffie− Hellman key agreement protocol*, RSA cryptosystem are developed, hence to find the condition on occurrence of such elements interesting. Even though we are not able to find exact primitive element, but there exists some methods where one can determine some conditions to find a primitive element in context of another.

Since we are discussing about existence of a primitive element in context of another one we begin with consideration of pairs $(\alpha, f(\alpha))$ where $\alpha(\neq 0) \in \mathbb{F}_q$ and $f(x) \in \mathbb{F}_q[x]$ be a rational expression of the previous one. If, for example, $f(x) = 1/x$, then $f(\alpha)$ is primitive in $\mathbb{F}_q$ whatever the α be ,but,this is not happening in all the cases. If we consider $f(x) = x + 1$ in $\mathbb{F}_2$ then 1 is a primitive element of $\mathbb{F}_2$ but $f(1) = 0$ is certainly not primitive in $\mathbb{F}_2$. Hence the role of finding the conditions come into play such that $(\alpha, f(\alpha))$ is a primitive pair where $f(x) = x + 1$ is also primitive.

Furthermore, not only this polynomial or function, but also rational functions like $f(x) = x + \frac{1}{x}$ also can be considered. Various developments have been made in this area but we are mainly focusing on this two basic functions i.e, to finding the conditions for the existence of consecutive primitive pair or to find the primitive pair of the form $\alpha, f(\alpha)$ where $f(x) = x + 1$ and the other one is finding such primitive pairs where $f(x) = x + \frac{1}{x}$.

3 Consecutive Primitive Elements

This word is done by S.D.Cohen in "Consecutive primitive roots in a finite field" [1] , where he focused on the conditions of existence of primitive root say α such that $\alpha + 1$ is also primitive on $\mathbb{F}_q$. It can be seen that 2, 3 and 7 are not such type of q. In his work, he found a sufficient method for proving that every other prime power can be considered as such q. For our convenience, we denote by F^* the subset of F containing q for which, there exists consecutive primitive roots .

We have the stranded result provided by E. Vegh for odd prime $q = p$ as follows:

Theorem 3.1. *For* $p \equiv 1(mod4)$*, if* $\theta(p-1) > \frac{1}{4}$*, then* $p \in F$.

Theorem 3.2. *For* $p \equiv 11(mod12)$*, if* $\theta(p-1) > \frac{1}{3}$ *, then* $p \in F$.

Where θ is defined as $\theta(m) = \frac{\phi(m)}{m}$, for any positive integer m.

Based on the above results, S.D.Cohen established the following theorems. Which are helpful for generalising the results.

Theorem 3.3. *For $q(>3) \in F$ for $q \not\equiv 7(mod12)$ and $q \not\equiv 1(mod60)$.*

But for even values of q, technically it is no more difficult to establish more generalised theorem.

Theorem 3.4. *For $q(>4)$ even,then every element of $\mathbb{F}_q$ can be expressed as sum of two primitive elements, that is, in this case $q \in F^*$.*

. **Some estimates of the above theorems** Let $\alpha \in \mathbb{F}_q^*$. Considering two divisors d_l, d_2 of $q-1$. Then we use $\mathfrak{N}(d_l, d_2)$ the number of elements $\beta(\neq 0, -\alpha)$ in $\mathbb{F}_q$ such that the gcd $(\omega(\beta), d_l) = 1$, $(\omega(\beta+\alpha), d_2) = 1$, where $\omega(\beta)$ be the divisor of $q-1$ such that $(q-1)/\omega(\beta)$ is the order of β in $\mathbb{F}_q^*$. Main objective of the estimation is to find condition such that $\mathfrak{N}(q-1, q-1) > 0$. Then for this estimation the following lemmas are established.

Lemma 3.5. $\mathfrak{N}(q-1, q-1) > \mathfrak{N}(d_l, q-1) + \mathfrak{N}(q-1, d_2) - \mathfrak{N}(d_l, d_2)$.

Lemma 3.6. *For, $e|q-1$, $\mathfrak{N}(1, e) = \theta(d)(q-1) - \psi_\beta(d), \mathfrak{N}(d, 1) = \theta(d)(q-1) - \psi_{-\beta}(d)$.*

Where $$\psi_\beta(d) = \begin{cases} 1 \; if (\omega(\beta), d) = 1 \\ 0 \; otherwise \end{cases}$$

Furthermore, for $\alpha = 1$ and $d > 1$, we have $\mathfrak{N}(1, d) = \theta(d)(q-1)$

and $$\mathfrak{N}(d, 1) = \begin{cases} \frac{1}{2}(q-3) \; if q \equiv 3 \, mod(4) \; and d = 2 \\ \theta(d)(q-1) \; otherwise \end{cases}$$

For odd values of q, the following results are given .

Lemma 3.7. $$\mathfrak{N}(d, 2) = \begin{cases} \frac{1}{4}(q-3) \; if q \equiv 3 \, mod(4) \; and d = 2 \\ \frac{1}{2}\theta(d)(q-1) \; otherwise \end{cases}$$

where q odd, $\alpha = 1$ and d an even divisor of $q-1$.
Furthermore, for $q \equiv 1(mod4)$, $\mathfrak{N}(2, d) = \frac{1}{2}\theta(d)(q-1)$.

Lemma 3.8. $\mathfrak{N}(q-1, q-1) > (\theta(q-1) - \frac{1}{4})(q-1)$, *if $q \equiv 1(mod4)$ and $\alpha = 1$.*
Again, if $\theta(q-1) > 1$, then $q \in F$.

Lemma 3.9. $\mathfrak{N}(q-1, q-1) > \frac{1}{2}(3\theta(q-1) - 1)(q-1)$, *if $q \equiv 3(mod4)$ and $q > 3$. Particularly if $\theta(q-1) > 1$, then $q \in F$.*

Based on the above theorems, the following results were obtained by Cohen in [1].

main theorems

Theorem 3.10. *This theorem is for q odd, where we use the notation $W(n) = 2^w(n)$, where $w(n)$ denotes the number of distinct prime divisors of positive integer n. If one of the following satisfies , then $\mathbb{F}_q$ has pair of consecutive primitive elements.*

(i) $W(q-1) > 16$ *or* $q > 1.16 \times 10^18$.

(ii $W(q-1) > 14$ *or* $q > 4.51 \times 10^15$, *where q is odd positive integer such that $q \not\equiv 1$ mod(3).*

(iii) $W(q-1) > 13$ *or* $q > 2.82 \times 10^14$, *where q is odd positive integer such that* $q \equiv 1$ *mod(12) and* $q \not\equiv 1$ *mod(60).*

Theorem 3.11. *For q even prime power i.e,* $q = 2^k$, *for some positive integer k, the results established as follows, where the notations used same as above.*

(i) $\mathbb{F}_q$ *has consecutive primitive elements if* $k \geq 4W(q-1)$.

(ii) For $k > 12$, $\mathbb{F}_q$ *has pair of primitive elements.*

4 Primitive elements of special form $\alpha + \alpha^{-1}$, when α is also primitive

For this result, a very well known "Lenstra-Schroof" has been used. To use this method we need some more informations about "Character of finite field", and some other useful theorems to complete the result. Hence we begin this section by some definitions and proceed to various theorems used in this method or process.

Definition 4.1. ***Character*** *For a finite abelian group G and* $\mathbb{C}^* := \{z \in \mathbb{C} : |z| = 1\}$ *be the multiplicative group of all complex numbers with modulus 1 . A character* χ *of G can be defined as a homomorphism from G into the group* $\mathbb{C}^*$, *which is defined as* $\chi(a_1a_2) = \chi(a_1)\chi(a_2)$ *for all* $a_1, a_2 \in G$. *The characters of G also forms a group under multiplication called dual group or character group of G which is denoted by* $\widehat{G}$. *From the definition it is clear that* $\widehat{G}$ *is isomorphic to G. Again the trivial character denoted by* χ_0 *is defined as* $\chi_0(a) = 1$ *for all* $a \in G$.

It is well known that a finite field $\mathbb{F}_{q^n}$ *has two types of abelian groups, such as additive group* $\mathbb{F}_{q^n}$ *and multiplicative group* $\mathbb{F}^*_{q^n}$. *Hence, there exist two types of characters for finite field* $\mathbb{F}_{q^n}$, *such that additive character for* $\mathbb{F}_{q^n}$ *and multiplicative character for* $\mathbb{F}^*_{q^n}$. *Multiplicative characters are used in extended from* $\mathbb{F}^*_{q^n}$ *to* $\mathbb{F}_{q^n}$ *by applying the conditions*

$$\chi(0) = \begin{cases} 0 \text{ if} \chi \neq \chi_0 \\ 1 \text{ if} \chi = \chi_0 \end{cases}$$

Since $\widehat{\mathbb{F}^*_{q^n}} \cong \mathbb{F}^*_{q^n}$, *so* $\widehat{\mathbb{F}^*_{q^n}}$ *is cyclic and for any divisor d of* q^n-1, *there are exactly* $\phi(d)$ *characters of order d in* $\widehat{\mathbb{F}^*_{q^n}}$.

Definition 4.2. ***e-free element*** *For a divisor k of* $q^n - 1$, *an* $k-free$ *element x of* $\mathbb{F}_q$ *is defined as for any* $h|k, x = y^d$ *where* $y \in \mathbb{F}_{q^n}$ *gives* $h = 1$ *i.e, if* $gcd(h, \frac{q^n-1}{ord_{q^n}(x)}) = 1$. *Hence an element* $q^n - 1$*-free, then it is primitive. Form this on, we are going to focus on the* $q^n - 1$*-free elements instead of primitive element.*

Definition 4.3. ***Character function*** *The character function for the subset of k-free elements of* $\mathbb{F}^*_{q^n}$ *is defined by Cohen and Huczynska[2, 3] as follows*

$$\varrho_k : \alpha \mapsto \Delta(k)\sum_{h|k}(\frac{\mu(h)}{\phi(h)}\sum_{\chi_h}\chi_h(\alpha))$$

where $\Delta(k) := \frac{\phi(k)}{k}$, μ *be the Möbius function and* χ_d *denotes multiplicative character of order h.*

Theorems used in the main result

Theorem 4.4. ***([5], Theorem 5.4)*** $\sum_{\alpha \in G}\chi(\alpha) = 0$ *and* $\sum_{\chi \in \widehat{G}}\chi(\alpha) = 0$. *Where* χ *is any nontrivial character and* α *any nontrivial element of finite abelian group G.*

Theorem 4.5. ***([6], theorem 5.11)*** *Let χ be a nontrivial multiplicative character and ψ a nontrivial additive character of $\mathbb{F}_{q^n}$. Then*

$$|\sum_{\alpha\in\mathbb{F}^*_{q^n}} \chi(\alpha)\psi(\alpha)| = q^{n/2}.$$

Theorem 4.6. ***([6], Corollary 2.3.***

For two nontrivial multiplicative characters χ_1, χ_2 of the finite field $\mathbb{F}_q$. Let $g_1(x)$ and $g_2(x)$ be two monic polynomials which are pairwise coprime in $\mathbb{F}_q[x]$, in which $f_i(x)$ can not be of the form $h(x)^{ord(\chi_i)}$ for $i = 1, 2$; $h(x) \in \mathbb{F}_{q^n}[x]$ with degree at least 1. Then

$$|\sum_{x\in\mathbb{F}_q} \chi_1(g_1(x))\chi_2(g_2(x))| \leq (n_1 + n_2 - 1)q^{1/2}$$

where n_1 and n_2 are the degrees of largest square free divisors of g_1 and g_2 respectively.

Theorem 4.7. ***([5], Theorem 5.41)***

For a multiplicative character χ of $\mathbb{F}_{q^n}$ of order $k > 1$ and $g \in \mathbb{F}_{q^n}[x]$ be a monic polynomial of positive degree such that, it is not an k^{th} power of a polynomial over $\mathbb{F}_{q^n}$. Let h be the number of distinct roots of g in its splitting field over $\mathbb{F}_{q^n}$. Then for every $a \in \mathbb{F}_{q^n}$, we have

$$|\sum_{\alpha\in\mathbb{F}_{q^n}} \chi(ag(\alpha))| \leq (h - 1)q^{n/2}$$

Lemma 4.8. ***([4], Lemma 2.6)***

Let $n > 1, d > 1$ be integers and Γ be the set of primes $\leq$ d. Then the set defined as $\mathfrak{L} = \prod_{\alpha\in\Gamma} r$. Consider that every prime factor $a < d$ of n is in the set Γ. Then

$$W(n) \leq \frac{\log n - \log \mathfrak{L}}{\log d} + |\Gamma| \tag{4.1}$$

Taking m a positive integer and p_m is to denote the m^{th} prime. If $d = p_m$, then Γ becomes the set of primes such that the primes are less than p_m , $|\Gamma| = m$ i.e., and hence the inequality (4.1) becomes

$$W(n) \leq \frac{\log n - \sum_{i=1}^{m} \log\, p_i}{\log p_m} + m \tag{4.2}$$

Main theorem

After using all the results above Liao, Li and Pu established the following result in [1]. They are using the "Lenstra-Schroof" method to find the results.

Theorem 4.9. *The necessary condition for the existence of the primitive element of the form $\alpha + \alpha^(- 1)$ in finite field is $\mathbb{F}_q$, where gcd(q, n)=1 is $q^{\frac{n}{2}} > 2^\omega$. Here ω denotes the number of distinct prime divisors of $q^n - 1$.*

By applying this theorem, one can establish that for $n > 13$ and $k \geq 4$, then $\mathbb{F}_{q^n}$ has such primitive pair.

5 Conclusion

From the above results we come to the conclusion that the above results that even though determining the primitive elements is not possible yet, but the problem of existence of primitive roots in context of one-another is opening new windows of opportunities in this area of finite field. By taking different rational expressions, with the same process one can obtain further results, which will be very helpful to apply in different areas of finite field, coding theory and cryptography.

6 Bibliography

[1] S.D.Cohen, Consecutive primitive roots in a finite field, *Proc. Amer. Math. Soc.*, **93**(2) (1985) 189-197.

[2] S.D.Cohen and S.Huczynska, The primitive normal basis theorem without a computer, *J. Lond. Math. Soc.* **67**(1) (2003) 41-56.

[3] S.D.Cohen and S.Huczynska, The strong primitive normal basis theorem, *Acta. Arith.* **143**(4) (2010) 299-332.

[4] Qunying Liao, Jiyou Li, Keli Pu, On the Existence for Some Special Primitive Elements in Finite Fields, *Chin. Ann. Math.* **37** (2) (2016) 259-266.

[5] R.Lidl and H.Niederreiter, *Finite Fields*, 2nd edn. (Cambridge University Press, Cambridge, 1997).

[6] D.Wan, Generators and irreducible polynomials over finite fields, *Math. Comp.*, **66**(219) (1997) 1195-1212

Congruences for Apéry and Apéry like numbers: a brief survey

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Abstract. In [2], R. Apéry while proving the irrationality of $\zeta(2)$ and $\zeta(3)$ introduced the numbers A_n and B_n, defined respectively as

$$A_n = \sum_{k=0}^{n} \binom{n+k}{k}\binom{n}{k}^2 \text{ and } B_n = \sum_{k=0}^{n} \binom{n+k}{k}^2\binom{n}{k}^2.$$

These numbers were found to satisfy many interesting properties. In this survey we shall discuss numerous works done in this direction including congruences satisfied by them and generalized Apéry-like numbers.

Keywords. Apéry numbers, Gaussian hypergeometric series, Supercongruences.

1 Introduction

In the year 1979, R. Apéry[2] while proving the irrationality of $\zeta(2) = \sum_{n=0}^{\infty} \frac{1}{n^2}$ and $\zeta(3) = \sum_{n=0}^{\infty} \frac{1}{n^3}$ introduced the numbers A_n and B_n, given by

$$A_n = \sum_{k=0}^{n} \binom{n+k}{k}\binom{n}{k}^2 \text{ and } B_n = \sum_{k=0}^{n} \binom{n+k}{k}^2\binom{n}{k}^2.$$

The Apéry number A_n rises as a solution of the recursion formula

$$(n+1)^2u_{n+1} = (an^2+an+b)u_n - cn^2u_{n-1}; \quad u_{-1} = 0, u_0 = 1, \tag{1}$$

for $(a,b,c) = (11,3,-1)$. In [19], Zagier discovered other values of the parameters (a,b,c) for which (1) has an integral solution. As a result, he discovered six sporadic solutions apart from other solutions. With the choice of parameters $(a,b,c,d) = (17,5,1,0)$, the sequence B_n satisfy another three-term recurrence

$$(n+1)^3 u_{n+1} = (2n+1)(an^2+an+b)u_n - n(cn^2+d)u_{n-1}; \quad u_{-1}=0, u_0=1 \tag{2}$$

in the similar way as A_n do in (1). For $d=0$, Almkvist & Zudilin [1] performed a systematic computer search to find six sporadic solutions for (2) in addition to other integral solutions. In the similar way Cooper [8] introduced an additional parameter d and further obtained three more sporadic solutions.

The integral solutions of (1) and (2) for particular values of the parameters obtained by Zagier [19], Almkvist & Zudilin [1] and Cooper [8], are named as Apéry-like numbers as these numbers enjoy many of the remarkable properties of the Apéry numbers, such as satisfying Lucas congruences, connections to modular forms, and supercongruences. The integral solutions to both the recurrence relations are listed in Table 1 and Table 2.

	(a,b,c)	$A(n)$
(a)	$(7,2,-1)$	$\sum_k \binom{n}{k}^3$
(b)	$(11,3,-1)$	$\sum_k \binom{n}{k}^2 \binom{n+k}{k}$
(c)	$(10,3,9)$	$\sum_k \binom{n}{k}^2 \binom{2k}{k}$
(d)	$(12,4,32)$	$\sum_k \binom{n}{k}\binom{2k}{k}\binom{2(n-k)}{n-k}$
(f)	$(9,3,27)$	$\sum_k (-1)^k 3^{n-3k} \binom{n}{3k} \frac{(3k)!}{k!^3}$
(g)	$(17,6,72)$	$\sum_{k,l} (-1)^k 8^{n-k} \binom{n}{k}\binom{k}{l}^3$

Table 0.1: Sporadic solutions of the recurrence relation (1)

Along with certain congruence properties, these numbers also holds relation to hypergeometric functions and modular forms as well. In this article we shall discuss the research in this direction.

2 Congruences satisfied by Apery and Apéry-like numbers

Since the introduction of Apéry numbers, mathematicians started looking more closely at these numbers and found that they satisfy many interesting properties. Some of the important results concerning periodic and congruence properties satisfied by the Apéry numbers are discussed briefly in this section.

Chowla, Cowles, and Cowles [7] were first to study properties of Apéry numbers. They proved many basic congruences for the Apéry numbers together with the fact that B_n is odd.

Proposition 2.1 ([7]). *For $n \geq 0$, $B_{5n+1} \equiv 0 \pmod 5$ and $B_{5n+3} \equiv 0 \pmod 5$.*

Theorem 2.2 ([7]). *For all primes p, $B_p \equiv 5 \pmod{p^2}$.*

	(a,b,c,d)	$B(n)$
(δ)	$(7,3,81,0)$	$\sum_k(-1)^k 3^{n-3k}\binom{n}{3k}\binom{n+k}{n}\frac{(3k)!}{k!^3}$
(η)	$(11,5,125,0)$	$\sum_{k=0}(-1)^k\binom{n}{k}^3\left(\binom{4n-5k-1}{3n}+\binom{4n-5k}{3n}\right)$
(α)	$(10,4,64,0)$	$\sum_k\binom{n}{k}^2\binom{2k}{k}\binom{2(n-k)}{n-k}$
(ϵ)	$(12,4,16,0)$	$\sum_k\binom{n}{k}^2\binom{2k}{n}^2$
(ζ)	$(9,3,-27,0)$	$\sum_{k,l}\binom{n}{k}^2\binom{n}{l}\binom{k}{l}\binom{k+l}{n}$
(γ)	$(17,5,1,0)$	$\sum_k\binom{n}{k}^2\binom{n+k}{k}^2$
(s_7)	$(13,4,-27,3)$	$\sum_k\binom{n}{k}^2\binom{n+k}{k}\binom{2k}{n}$
(s_{10})	$(6,2,-64,4)$	$\sum_k\binom{n}{k}^4$
(s_{18})	$(14,6,192,-12)$	$\sum_{k=0}\binom{n}{k}\binom{2k}{k}\binom{2(n-k)}{n-k}\left[\binom{2n-3k-1}{n}+\binom{2n-3k}{n}\right]$

Table 0.2: Sporadic solutions of the recurrence relation (2)

In addition, they proposed that the Apéry numbers satisfy the following properties.

Theorem 2.3 ([7])**.** *For $n \geq 0$ and $p \geq 5$,*

$$
\begin{aligned}
&(i)\ B_{2n} \equiv 1 \pmod 8.\\
&(ii)\ B_{2n+1} \equiv 5 \pmod 8.\\
&(iii)\ B_{2n} \equiv 1 \pmod 3.\\
&(iv)\ B_{2n+1} \equiv 2 \pmod 3.\\
&(v)\ B_p \equiv 5 \pmod{p^3}.\\
&(vi)\ B_{2n+1} \equiv 0 \pmod 5.
\end{aligned}
$$

I. Gessel [10] confirmed almost all congruences of Theorem 2.3, and proved that B_n satisfies the Lucas congruence i.e., if $n = n_0 + n_1 p + \cdots + n_r p^r$ is the expansion of n in base p, then

$$B_n \equiv B_{n_0} B_{n_1} \cdots B_{n_r} \pmod p.$$

Moreover, he proved that for primes $p \geq 5$,

$$B_{pn} \equiv B_n \pmod{p^3}. \tag{3}$$

Based on the Lucas property of Apéry numbers, Gessel examined periodic properties of the Apéry numbers in modulo 8 and 9.

Theorem 2.4 ([10], Theorem 3)**.**

$$
\begin{aligned}
&(i)\ B_n \equiv 5^n \pmod 8.\\
&(ii)\ B_{n_0+3n_1+\cdots+3^s n^s} \equiv B_{n_0} B_{n_1} \cdots B_{n_s} \pmod 9,\ \textit{where}\ \ 0 \leq n_i < 3.
\end{aligned}
$$

He further posed a problem on periodicity of these numbers in modulo 16, which is recently solved by E. Rowland and R. Yassawi [18].

Extending certain results of Mimura [13], F. Beukers [3] gave the following result.

Theorem 2.5 ([3], Theorem 1)**.** *Let $m, r \in \mathbb{N}$ and $p > 3$. Then*

$$A_{mp^r-1} \equiv A_{mp^{r-1}-1} \pmod{p^{3r}} \tag{4}$$

$$B_{mp^r-1} \equiv B_{mp^{r-1}-1} \pmod{p^{3r}}. \tag{5}$$

Recently, Malik and Straub [12] proved that the Lucas congruence is satisfied by all the sporadic solutions of recursions (1) and (2) listed in Table 1 and 2. As an application, they deduced certain periodic properties for those numbers.

Definition 2.6 (Periodic sequences)**.** *A sequence $C(n)$ is eventually periodic if there exists an integer $M > 0$ such that $C(n+m) = C(n)$ for all sufficiently large n.*

Corollary 2.7 ([12], Corollary 5.1)**.** *None of the sequences from Tables 1 and 2 is eventually periodic modulo p for any prime $p > 5$.*

For periodicity of the numbers arising from the recurrence relation (1) and (2), they gave the following result.

Corollary 2.8 ([12], Corollary 5.2)**.** *Let $C(n)$ be any sequence from Tables 1 and 2.*

- $C(n) \equiv C(1) \pmod 2$ *for all* $n \geq 1$.
- $C(n) \equiv C(1) \pmod 3$ *for all* $n \geq 1$ *if* $C(n)$ *is one of* $(c), (f), (g), (\delta), (\alpha), (\epsilon), (\zeta), s_{18}$, *and* $C(n) \equiv (-1)^n \pmod 3$ *for all* $n \geq 0$ *if* $C(n)$ *is* (a) *or* (γ).
- $C(n) \equiv 3^n \pmod 5$ *for all* $n \geq 0$ *if* $C(n)$ *is* (b), *and* $C(n) \equiv 0 \pmod 5$ *for all* $n \geq 1$ *if* $C(n)$ *is* (η).

3 Congruences for generalization of Apéry-like numbers

The development of Apéry numbers and their related congruences motivated mathematicians to introduce mathematically riched other number theoretical objects exhibiting similar types of properties.

M. Coster [9] considered a generalization of Apéry numbers given by the formula

$$A(n, m, l, \lambda) := \sum_{k=0}^{n} \binom{n}{k}^m \binom{n+k}{k}^l \lambda^k,$$

and proved that for $m, l \in \mathbb{N}, \lambda = \pm 1$ and $p \geq 5$,

$$A(sp^r, m, l, \lambda) \equiv A(sp^{r-1}, m, l, \lambda) \pmod{p^{3r}} \text{ if } m \geq 3, \tag{6}$$

$$A(sp^r - 1, m, l, \lambda) \equiv A(sp^{r-1} - 1, m, l, \lambda) \pmod{p^{3r}} \text{ if } l \geq 3.$$

Note that the above congruences are generalizations of (3), (4), and (5) on Apéry numbers. Recently, Krattenthaler and Möller [11] characterized the modular behaviour of these generalised Apéry numbers in modulo 9.

The identification of Apéry numbers as the coefficients of certain power series motivated Chan, Cooper, and Sica [5] to obtain certain Apéry-like sequences including the Domb numbers

$$\beta_n = \sum_{j=0}^{n} \binom{n}{j}^2 \binom{2j}{j} \binom{2(n-j)}{n-j}.$$

They further proved that (3) is satisfied by some of them, and for other the relation is posed as conjectures. Two of their conjectures were proved by Chan, Kontogeorgis, Krattenthaler and Osburn [6] using combinatorial properties of the power series expansion of modular forms. Osburn and Sahu [17] proved other two congruences using theory of modular forms while the remaining three conjectures are still open.

Osburn-Sahu [15, 16] considered generalization of Apéry-like numbers (ϵ) and (α) in Table 2 as

$$C(n, A, B) := \sum_{k=0}^{n} \binom{n}{k}^A \binom{2k}{k}^B, \text{ and } D(n, A, B, C) := \sum_{k=0}^{n} \binom{n}{k}^A \binom{2k}{k}^B \binom{2(n-k)}{n-k}^C$$

and deduced that these two generalized sequences also satisfy (6). Furthermore, Osburn-Sahu-Straub [14] considered generalization of s_7 in Table 2 as

$$S(n, A, B, C) := \sum_{k=0}^{n} \binom{n}{k}^A \binom{n+k}{k}^B \binom{2k}{n}^C$$

and proved that (6) is followed by these numbers for $A \geq 2$ and $B, C \geq 0$.

4 Conclusion

The importance of Apéry number congruences lies in the fact that the relations satisfied by them are akin to Akin Swinnerton-Dyer congruences for special congruence subgroups. For details see [4]. Along with the discussed properties, the Apéry numbers has also connections to other mathematical objects, including modular froms and hypergeometric functions.

5 Bibliography

[1] G. Almkvist and W. Zudilin, *Differential equations, mirror maps and zeta values*, In: Mirror symmetry. V, volume 38 of AMS/IP Stud. Adv. Math., pp. 481–515. Amer. Math. Soc., Providence, RI, (2006).

[2] R. Apéry, *Irrationalite de* $\zeta(2)$ *and* $\zeta(3)$, Asterisque **61** (1979), 11–13.

[3] F. Beukers, *Some congruences for Apéry numbers*, J. Number Theory **21** (1985), 141–155.

[4] F. Beukers, *Consequences of Apéry's work on* $\zeta(3)$, [A report by F. Beukers] (2003), 1–6.

[5] H. H. Chan, S. Cooper and F. Sica, *Congruences satisfied by Apéry-like numbers*, Int. J. Number Theory **6** (2010), no. 1,89–97.

[6] H. H. Chan, A. Kontogeorgis, C. Krattenthaler and R. Osburn, *Supercongruences satisfied by coefficients of* $_2F_1$ *hypergeometric series* Annales des sciences mathÃƒÂ©matiques du Quabec **34**, (2010), no. 1, 25-36.

[7] S. Chowla, J. Cowles, and M. Cowles, *Congruence properties of Apéry numbers*, J. Number Theory **12** (1980), 188–190.

[8] S. Cooper, *Sporadic sequences, modular forms and new series for* $1/\pi$, Ramanujan J. **29**(13), (2012), 163–183.

[9] M. Coster, *Supercongruences*, Ph.D. thesis, Universiteit Leiden, 1988.

[10] I. Gessel, *Some congruences for Apéry numbers*, J. Number Theory **14** (1982), 362–368.

[11] C. Krattenthaler, and T. W. Mueller, *Generalised Apéry numbers modulo 9*, Journal of Number Theory **147** (2015), 708-720.

[12] A. Malik and A. Straub, *Divisibility properties of sporadic Apéry-like numbers*, Research in Number Theory, **2**:5, (2016).

[13] Y. Mimura, *Congruence properties of Apéry numbers*, J. Number Theory **16** (1983), 138–146.

[14] Robert Osburn, Brundaban Sahu and Armin Straub, *Supercongruences for Sporadic Sequences*, Proceedings of the Edinburgh Mathematical Society **59** (2016), 503-518.

[15] R. Osburn and B. Sahu, *Supercongruences for Apéry-like numbers*, Advances in Applied Mathematics **47** (2011), 631-638.

[16] R. Osburn and B. Sahu, *A supercongruence for generalized Domb numbers*,Functiones et Approximatio Commentarii Mathematici, **48**, no.1,(2013), 29-36.

[17] R. Osburn and B. Sahu, *Congruences via modular forms*, Proceedings of the American Mathematical Society **47** (2010),2375-2381.

[18] E. Rowland and R. Yassawi, *Automatic congruences for diagonals of rational functions*, J.de Théorie des Nombres de Bordeaux **27**(1), (2015), 245-288.

[19] Zagier, D: Integral solutions of Apery-like recurrences. In: Harnad, J, Winternitz, P (eds.) Groups and symmetries. From Neolithic to John McKay, volume 47. American Mathematical Society, Providence, RI, (2009).

Inequalities on Ranks and Cranks of Partitions

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Abstract. In this chapter, we give an introduction to ranks and cranks of partitions. We also give a brief review of literature on the works done so far on inequalities involving rank and crank differences.

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Keywords. Partitions, Ranks, Cranks.

1 Introduction

A partition $\pi = (\pi_0, \pi_1, \ldots, \pi_{k-1})$ of a nonnegative integer n is a finite sequence of non-increasing positive integer *parts* $\pi_0, \pi_1, \ldots, \pi_{k-1}$ such that $\pi_0+\pi_1+\cdots+\pi_{k-1} = n$. The partition function $p(n)$ is defined as the number of partitions of n. For example, $p(5)$=7, since there are seven partitions of 5, namely,

$$5, 4+1, 3+2, 3+1+1, 2+2+1, 2+1+1+1, 1+1+1+1+1.$$

By convention, $p(0) = 1$. The generating function for $p(n)$, due to Euler, is given by

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty},$$

where, for any complex number a and q, with $|q| < 1$, we define

$$\begin{aligned}(a;q)_0 &:= 1,\\ (a;q)_n &:= \prod_{k=0}^{n-1}(1-aq^k), \quad n \geq 1\end{aligned}$$

and

$$(a;q)_\infty := \lim_{n\to\infty}(a;q)_n.$$

Ramanujan [11], found nice congruence properties for $p(n)$ modulo 5, 7, and 11, namely, for any nonnegative integer n,

$$p(5n+4) \equiv 0 \pmod 5, \tag{7}$$

$$p(7n+5) \equiv 0 \pmod 7, \tag{8}$$

and

$$p(11n+6) \equiv 0 \pmod{11}. \tag{9}$$

In 1944 Dyson [5] defined the rank of a partition as the largest part minus the number of parts. For example, the partition $5+4+3+1$ has rank $5-4=1$. Let $N(m,n)$ denote the number of partitions of n with rank m, then

$$\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} N(m,\ n)z^m q^n = \sum_{k=0}^{\infty}\frac{q^{k^2}}{(zq;\ q)_k(z^{-1}q;\ q)_k}.$$

Let $N(r,\ m,\ n)$ denote the number of partitions of n with rank congruent to r modulo m. In the same paper, he conjectured that

(i) $N(r,\ 5,\ 5n+4) = \dfrac{1}{5}\ p(5n+4),\ 0 \le r \le 4,$

(ii) $N(r,\ 7,\ 7n+5) = \dfrac{1}{7}\ p(7n+5),\ 0 \le r \le 6.$

It is easy to see that the above two conjectures imply Ramanujan's congruences (7) and (8) respectively. These conjectures were subsequently proved by Atkin and Swinnerton-Dyer [3] in 1954. However Dyson's rank did not separate the partition of $11n+6$ into 11 equal classes even though Ramanujan's congruence (9) holds. So, he conjectured an analogue of rank, called crank, which would be able to imply all the three congruences. After forty four years, Andrews and Garvan [1] defined the crank of a partition as

$$\text{crank}(\pi) := \begin{cases} \pi_0, & \text{if } \mu(\pi)=0, \\ \nu(\pi)-\mu(\pi), & \text{if } \mu(\pi)>0, \end{cases}$$

where, $\mu(\pi)$ denotes the number of ones in π and $\nu(\pi)$ denotes the number of parts of π larger than $\mu(\pi)$. For example, the partition $5+4+3+1$ has crank $3-1=2$. The crank provided combinatorial interpretation of all the three congruences (7)–(9).

Let $M(m,n)$ denote the number of partitions of n with crank m, then

$$\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} M(m,\ n)z^m q^n = \frac{(q;\ q)_\infty}{(zq;\ q)_\infty(z^{-1}q;\ q)_\infty}.$$

Let $M(r,\ m,\ n)$ denote the number of partitions of n with crank congruent to r modulo m. In the same paper, they provided the following interpretations

(i) $M(r,\ 5,\ 5n+4) = \dfrac{1}{5}\ p(5n+4),\ 0 \le r \le 4,$

(ii) $M(r,\ 7,\ 7n+5) = \dfrac{1}{7}\ p(7n+5),\ 0 \le r \le 6,$

(iii) $M(r,\ 11,\ 11n+6) = \dfrac{1}{11}\ p(11n+6),\ 0 \le r \le 10.$

2 Literature Review

Lewis [10] gave a combinatorial proof to show that

(i) $N(0,\ 2,\ 2n) < N(1,\ 2,\ 2n)$ and

(ii) $N(1,\ 2,\ 2n+1) < N(0,\ 2,\ 2n+1)$.

Andrews and Lewis [2] proved several inequalities as given in the following theorems.

Theorem 2.1. *For all $n \ge 0$*

(i) $M(0,\ 2,\ 2n) > M(1,\ 2,\ 2n)$ *and*

(ii) $M(1,\ 2,\ 2n+1) > M(0,\ 2,\ 2n+1)$.

Theorem 2.2. *For $n = 2, 8, 10$ and 26,*

$$N(0,4,n) = N(2,4,n),$$

while, for all other n,

$$\begin{aligned} N(0,4,n) &> N(2,4,n)\ if\ n \equiv 0,1 \pmod 4, \\ N(0,4,n) &< N(2,4,n)\ if\ n \equiv 2,3 \pmod 4. \end{aligned}$$

Theorem 2.3. *(i) For $n \ne 1$, $M(0,4,2n) > M(1,4,2n)$,*

(ii) For $n \ne 2$, $M(0,4,2n-1) < M(1,4,2n-1)$,

(iii) For $n > 0$, $M(2,4,2n) > M(1,4,2n)$,

(iv) For $n > 0$, $M(2,4,2n-1) < M(1,4,2n-1)$.

Theorem 2.4. *(i) For all $n \ge 1$, $N(0,4,2n) < N(1,4,2n)$,*

(ii) For all $n \ge 1$, $N(0,4,2n-1) > N(1,4,2n-1)$,

(iii) For all $n \ge 1$, $N(2,4,2n) < N(1,4,2n)$,

(iv) For all $n \ge 2$, $N(2,4,2n-1) > N(1,4,2n-1)$.

They also showed that

$$\sum_{n\geq 0}\{M(0,3,n)-M(1,3,n)\}\,q^n=\frac{(q;q)_\infty^2}{(q^3;q^3)_\infty}$$

and

$$\sum_{n\geq 0}\{M(0,4,n)-M(2,4,n)\}\,q^n=\frac{(q;q)_\infty(q^2;q^2)_\infty}{(q^4;q^4)_\infty}$$

and made the following conjectures.

Conjecture 2.5. *For all $n>0$*

$$N(0,3,3n)<N(1,3,3n), \tag{10}$$
$$N(0,3,3n+1)>N(1,3,3n+1), \tag{11}$$
$$N(0,3,3n+2)<N(1,3,3n+2). \tag{12}$$

Conjecture 2.6. *For all n,*

$$M(0,3,3n)>M(1,3,3n), \tag{13}$$
$$M(0,3,3n+1)<M(1,3,3n+1), \tag{14}$$
$$M(0,3,3n+2)\leq M(1,3,3n+2)\ if\ n\neq 1, \tag{15}$$

with strict inequality in (15) *if $n\neq 4,5$.*

Conjecture 2.7. *For $n\neq 5$,*

$$M(0,4,n)\geq M(2,4,n)\ if\ n\equiv 0,3 \pmod 4, \tag{16}$$
$$M(0,4,n)\leq M(2,4,n)\ if\ n\equiv 1,2 \pmod 4, \tag{17}$$

the inequalities being strict if $n\neq 11,15,21$.

By using the circle method Kane [6] proved the first conjecture, Chan [4] proved the second conjecture, and Kim [8] found a more general proof of (13)–(17).

Several other such congruences have been given by various mathematicians. For example, Kang [7] conjectures that the sign of a certain arithmetic function

$$N_{(6)}(n):=\sum_{r=0,\pm 1}N(r,6,n)-\sum_{r=3,\pm 2}N(r,6,n)$$

is alternating, which has been proved Kim and Nam [9] by using the circle method. Also, with the help of this result, they could find the following results.

Corollary 2.8. *Let $r_2(n)$ be the number of representations of n as a sum of two squares and $p(n)$ be the number of partitions of n. Then, for all integers $n>1$,*

$$\sum_{k=0}^{\lfloor n/3\rfloor}(1)^{nk}r_2(k)p(n-3k)>0.$$

Corollary 2.9. *For all positive integers $n > 1$,*

$$(1)^n \sum_{k=0}^{\lfloor n/3 \rfloor} c_3(n-3k)wpod(k) > 0,$$

where $c_t(n)$ is the number of t-core partitions of n and

$$wpod(n) := \sum_{\pi \in POD(n)} (1)^{\#_o(\pi)},$$

$POD(n)$ being the set of partitions of n with distinct odd parts and $\#_o(\pi)$, the number of odd parts in the partition π.

3 Conclusion

Most of the conjectures mentioned above have been proved by using the circle method. But it would be interesting to find the elementary proofs using Ramanujan's theta function identities and some identities involving the Rogers-Ramanujan continued fraction.

4 Bibliography

[1] Andrews, G. E. and Garvan, F. G. Dyson's crank of a partition. *Bulletin of the American Mathematical society*, **18(2)**:167–171, 1998.

[2] Andrews, G. E. and Lewis, R. The ranks and cranks of partitions moduli $2, 3$ and 4. *Journal of Number Theory*, **85(1)**:74–84, 2000.

[3] Atkin, A. O. L. and Swinnerton-Dyer, P. Some properties of partitions. *Proceedings of the London Mathematical Society (3)*, **4**:84–106, 1954.

[4] Chan, O.-Y. Some asymptotics for cranks. *Acta Arithmetica*, **120**:107–143, 2005.

[5] Dyson, F. J. Some guesses in the theory of partitions. *Eureka*, **8**:10–15, 1944.

[6] Kane, D. M. Resolution of a conjecture of Andrews and Lewis involving cranks of partitions. *Proceedings of the American Mathematical Society*, **132**:2247–2256, 2004.

[7] Kang, S.-Y. Mock Jacobi forms in basic hypergeometric series. *Compositio Mathematica*, **145**:553–565, 2009.

[8] Kim, B. Periodicity of signs of Fourier coefficients of eta quotients. *Journal of Mathematical Analysis and Applications*, **385**:998–1004, 2012.

[9] Kim, B. and Nam, H. On a conjecture of Soon-Yi Kang on a certain partition rank difference. *The Ramanujan Journal*, **35**:467–477, 2014.

[10] Lewis, R. P. The ranks of partitions modulo 2. *Discrete Mathematics*, **167/168**:445–449, 1997.

[11] Ramanujan, S. Some properties of $p(n)$, the number of partitions of n. *Proceedings of the Cambridge Philosophical Society*, **19**:207–210, (1919).

A review on ℓ-Regular Partition Function

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Abstract. Study of different partition function with some certain restriction and proving ramanujan type congruences has become one of the rich research topic of recent times. Here, in this paper, we have done a literature review on ℓ-regular partition

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Keywords. ℓ-regular partition; partition congruence; q-series identities; Ramanujan's theta-functions.

1 Introduction

The theory of partitions of numbers is an interesting branch of number theory.The concept of partitions was given by Leonard Euler in 18 th century. After Euler though, the theory of partition had been studied and discussed by many other prominent mathematicians like Gauss, Jacobi, Schur, McMahon and Andrews etc but the joint work of Ramanujan with Prof. G.H. Hardy made a revolutionary change in the field of partition theory of numbers. Ramanujan and Hardy invented circle method which gave the first approximations of the partition of numbers beyond 200.

A partition of a positive integer n is a non-increasing sequence of positive integers, called parts, whose sum equals n. The number of partitions of a positive integer n is denoted by $p(n)$. For convenience, we set $p(0) = 1$, which means it is considered that 0 has one partition. The generating function for the partition function is generally given by

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty} \tag{18}$$

where

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1 - aq^n) \tag{19}$$

.

In 1919, Ramanujan [Ram19], [Ram00, p. 210-213] established

$$p(5n+4) \equiv 0 \pmod{5}, \tag{20}$$

$$p(7n+5) \equiv 0 \pmod{7}, \tag{21}$$

$$p(11n+6) \equiv 0 \pmod{11} \tag{22}$$

Ramanujan's theta-functions identities are defined by

$$\phi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2 (q^2;q^2)_\infty,$$

$$\psi(q) := f(q,q^3) = \sum_{k=0}^{\infty} q^{k(k+1)/2} = (q^2;q^2)_\infty (q;q^2)_\infty,$$

and

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty,$$

where $f(a,b) = \sum_{k=-\infty}^{\infty} a^{k(k+1)/2} b^{k(k-1)/2}$, $|ab| < 1$ is the Ramanujan's general theta-function

Motivated by Ramanujan's congruences on $p(n)$ many other partition function are studied and Ramanujan type congruences are established by several mathematicians and researchers.One of the famous partition function is ℓ-regular partition. For any positive integer ℓ, ℓ-regular partition of a positive integer n is a partition of n such that none of its part is divisible by ℓ. For example,the number of 3-regular partition of 5 is 5 , namely

$$5, \quad 4+1, \quad 2+2+1, \quad 2+1+1+1, \quad 1+1+1+1+1.$$

If $b_\ell(n)$ denotes the number of ℓ-regular partition of n, then the generating function of $b_\ell(n)$ is given by

$$\sum_{n=0}^{\infty} b_l(n) q^n = \frac{(q^\ell;q^\ell_\infty}{(q;q)_\infty}. \tag{23}$$

2 Review of related literature for ℓ-regular partition function

In recent times study of partition function with some certain restrictions has become one of the popular research topic. The arithmetic properties of ℓ-regular partitions have been studied by many authors, for example see [AB16, BD15, CW14, CG13, DP09, HS10, Web11, XY14a, XY14b, Pen08] and references there in. Numerous congruences of the ℓ-regular partition function have been established in the spirit of Ramanujan by employing theta function identities and modular equations.

Andrews, Hirchhorn and Sellers [AHS10] proved some infinite family of congruences modulo 2 and 3 for $b_4(n)$. For example, for $\alpha \geq 1$ and $n \geq 0$,

$$b_4\big(3^{2\alpha+2}n + \frac{19 \cdot 3^{2\alpha+1} - 1}{8}\big) \equiv 0 \pmod{6}. \tag{24}$$

Hirchhorn and Sellers [HS10] obtained results for 5-regular partitions that are stronger than those obtained by Calkin et al. [CDJ08] They found infinitely many congruences for $b_5(n)$ and also they proved that $b_5(n)$ is even for at least 75% of the positive integers n. Webb [Web11] established an infinite family of congruences modulo 3 for $b_{13}(n)$. For example, for $\alpha \geq 2$ and $n \geq 0$,

$$b_{13}\big(3^\alpha n + \frac{5 \cdot 3^{\alpha-1} - 1}{2}\big) \equiv 0 \pmod 3. \tag{25}$$

Furcy and penniston [FP12] established some infinite families of congruences modulo 3 using theory of modular forms. For example,

$$b_7\big(3^{2\alpha+3} n + \frac{5 \cdot 3^{2\alpha+2} - 1}{4}\big) \equiv 0 \pmod 3. \tag{26}$$

Xia and Yao [XY14a] proved several infinite families of congruences modulo 2 for $b_9(n)$. For example, for $\alpha \geq 0$ and $n \geq 0$,

$$b_9\big(2^{6\alpha+4} n + \frac{5 \cdot 2^{6\alpha+3} - 1}{3}\big) \equiv 0 \pmod 2, \tag{27}$$

$$b_9\big(2^{6\alpha+7} n + \frac{2^{6\alpha+6} - 1}{3}\big) \equiv 0 \pmod 2. \tag{28}$$

Cui and Gu [CG13] also studied arithmetic properties of ℓ-regular partition function where $\ell = 2, 4, 5, 8, 13, 16$ and established some results by using p-dissections identities for Ramanujan's theta function $\psi(q)$ and $f(-q)$. For example, for $\alpha \geq 2$ and $n \geq 0$,

$$b_5\big(4 \cdot 5^{2\alpha+1} n + \frac{31 \cdot 5^{2\alpha} - 1}{6}\big) \equiv 0 \pmod 2. \tag{29}$$

For any odd prime p, $\alpha \geq 1$, and $n \geq 0$,

$$b_4\big(p^{2\alpha} n + \frac{(8i+p)p^{2\alpha} - 1}{8}\big) \equiv 0 \pmod 2. \tag{30}$$

Baruah and Das [BD15] proved some parity results for 7-regular and 23-regular partitions by employing Ramanujan's theta functions and their dissections. For example, if $r \in \{3, 4, 6\}$ and $s \in \{1, 5, 6\}$ then for all $n, \alpha \geq 0$, we have

$$b_7\big(2 \cdot 7^{2\alpha+1} n + 2r \cdot 7^{2\alpha} + \frac{5(7^{2\alpha} - 1)}{4} + 1\big) \equiv 0 \pmod 2, \tag{31}$$

$$b_7\big(2 \cdot 7^{2(\alpha+1)} n + 2s \cdot 7^{2\alpha+1} + \frac{21 \cdot 7^{2\alpha} - 1}{4}\big) \equiv 0 \pmod 2, \tag{32}$$

Baruah and Ahmed [AB16], proved several congruences for l-regular partitions for $\ell \in \{5, 6, 7, 49\}$ by using p-dissections of $(q;q)_\infty$, $\psi(q)$, $(q;q)_\infty^3$ and $\psi(q^2)(q;q)_\infty^2$. By using theory of modular forms, they also find alternative proofs of the congruences for 10- and 20- regular partitions which were earlier proved by Carlson and Webb [CW14]. For example, If p is a prime such that $p \equiv -1 \pmod 6$ then for all $\alpha \geq 0$

$$\sum_{n=0}^{\infty} b_5(25p^{2\alpha} n + \frac{25 \cdot p^{2\alpha} - 1}{6})q^n \equiv (-1)^{\alpha \cdot \frac{p-2}{3}} 5p^\alpha (q;q)_\infty^4 \pmod{25} \tag{33}$$

3 Concluding Remarks

As we have seen that there are so many Ramanujan's type congruences for ℓ-regular partition function with respect to certain modulo that have been proved by different mathematicians. Those results that are obtained by employing some modular equation or q-series identities which will help the researchers to do futher research in partition theory.

4 Bibliography

[AB16] Zakir Ahmed and Nayandeep Deka Baruah. New congruences for $\ell-$regular partitions for $\ell \in \{5, 6, 7, 49\}$. *The Ramanujan Journal*, 40(3):649–668, 2016.

[AHS10] George E Andrews, Michael D Hirschhorn, and James A Sellers. Arithmetic properties of partitions with even parts distinct. *The Ramanujan Journal*, 23(1-3):169–181, 2010.

[BD15] Nayandeep Deka Baruah and Kuwali Das. Parity results for 7-regular and 23-regular partitions. *Int. J. Number Theory*, 11:2221–2238, 2015.

[CDJ08] Neil Calkin, Nate Drake, Kevin James, Shirley Law, Philip Lee, David Penniston, and Jeanne Radder. Divisibility properties of the 5-regular and 13-regular partition functions. *Integers*, 8(2):A60, 2008.

[CG13] Su-Ping Cui and Nancy SS Gu. Arithmetic properties of ℓ-regular partitions. *Advances in Applied Mathematics*, 51(4):507–523, 2013.

[CW14] Rowland Carlson and John J Webb. Infinite families of infinite families of congruences for k-regular partitions. *The Ramanujan Journal*, 33(3):329–337, 2014.

[DP09] Brian Dandurand and David Penniston. ℓ-divisibility of ℓ-regular partition functions. *The Ramanujan Journal*, 19(1):63–70, 2009.

[FP12] David Furcy and David Penniston. Congruences for ℓ-regular partition functions modulo 3. *The Ramanujan Journal*, 27(1):101–108, 2012.

[HS10] Michael D Hirschhorn and James A Sellers. Elementary proofs of parity results for 5-regular partitions. *Bulletin of the Australian Mathematical Society*, 81(1):58–63, 2010.

[Pen08] David Penniston. Arithmetic of ℓ-regular partition functions. *International Journal of Number Theory*, 4(02):295–302, 2008.

[Ram19] Srinivasa Ramanujan. Some properties of p (n), the number of partitions of n. In *Proc. Cambridge Philos. Soc*, volume 19, pages 207–210, 1919.

[Ram00] S Ramanujan. Collected papers, cambridge university press, cambridge, 1927; reprinted by chelsea, new york, 1962; reprinted by the american mathematical society. 2000.

[Web11] John J Webb. Arithmetic of the 13-regular partition function modulo 3. *The Ramanujan Journal*, 25(1):49–56, 2011.

[XY14a] Ernest XW Xia and Olivia XM Yao. Parity results for 9-regular partitions. *The Ramanujan Journal*, 34(1):109–117, 2014.

[XY14b] Ernest XW Xia and Olivia XM Yao. A proof of keith's conjecture for 9-regular partitions modulo 3. *International Journal of Number Theory*, 10(03):669–674, 2014.

Almost Circular Balancing Numbers

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Abstract. In this short note, we define the concept of an almost circular balancing number and study some special cases of these numbers. We also connect their study with other types of balancing numbers.

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1 Introduction

A natural number n is called a *balancing number* with *balancer* r if

$$1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r).$$

The concept of balancing numbers were introduced by Behera and Panda in [1]. Since then, several people have studied these numbers and their modifications (see [5, 7, 8] and the references therein). One of these modifications is the concept of an *almost balancing number* introduced by Panda and Panda in [6], which is defined as below.

Definition 1.1. *A natural number n is called an almost balancing number if it satisfies the equation*

$$1 = |(n+1) + (n+2) + \cdots + (n+r) - \{1 + 2 + \cdots + (n-1)\}|,$$

for some natural number r called the almost balancer corresponding to n.

If

$$(n+1) + (n+2) + \cdots + (n+r) - \{1 + 2 + \cdots + (n-1)\} = 1$$

then n is called an A_1 balancing number, and if

$$(n+1) + (n+2) + \cdots + (n+r) - \{1 + 2 + \cdots + (n-1)\} = -1$$

then n is called an A_2 balancing number. Similar terminology is used for r.

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Recently, in [4], Panda and Panda defined circular balancing numbers as below.

Definition 1.2. *Let k be a fixed positive integer. We call a positive integer n, a k-circular balancing number if the Diophantine equation*

$$(k+1)+(k+2)+\cdots+(n-1)=(n+1)+(n+2)+\cdots+m+(1+2+\cdots+k-1)$$

holds for some natural number m.

Inspired by the recent work of Panda and Panda [6], we make the following definition.

Definition 1.3. *Let k be a fixed positive integer. We call a positive integer n, an almost k-circular balancing number if the Diophantine equation*

$$(k+1)+(k+2)+\cdots+(n-1)+1=(n+1)+(n+2)+\cdots+m+(1+2+\cdots+k-1)$$

holds for some natural number m.

We observe that, if $k=0$ we get the almost balancing numbers (also, 1-circular balancing numbers); and if $k=1$, we get the balancing numbers (also, 0-circular balancing numbers).

This paper is structured as follows: in Section 2 we motivate the study of almost circular balancing numbers, in Sections 3,4 and 5, we shall look into the cases $k=2,3$ and 4 in details, and in Section 6 remark for the general case. Our results are inspired by that of Panda and Panda [4] and have a similar flavour. Since the derivations are somewhat routine, we shall skip some details from the cases $k=3$ and 4. We shall close the paper with some general comments in Section 7 for future directions of study.

2 Motivation

Recently, Davala and Panda in [2], introduced the concept of a D-subbalancing and D-superbalancing numbers. They defined them as follows.

Definition 2.1. *For a positive integer D, we call a positive integer n, a D-subbalancing number if*

$$1+2+\cdots+(n-1)+D=(n+1)+(n+2)+\cdots+(n+r),$$

for some natural number r. If $D<0$, then such a number n is called a D-superbalancing number.

At the end of their paper, they posed the question about which values of D are feasible in their definition. In this regard, we give the following theorems.

Theorem 2.2. *If $D=k^2$, then the D-superbalancing numbers are the k circular balancing numbers.*

Proof. This follows from Definitions 1.2 and 2.1. □

Theorem 2.3. *If $D=k^2+1$, then the D-superbalancing numbers are the almost k circular balancing numbers.*

Proof. This follows from Definitions 1.3 and 2.1. □

Hence, it is interesting in the context of superbalancing numbers as well to study the circular and almost circular balancing numbers. In fact, the above and our results in Section 6 as well as results from Panda and Panda in [4] shows that there are infinitely many values of D for which there exists D-superbalancing numbers.

3 Almost 2 Circular Balancing Numbers

By the definition, a natural number x is an almost 2-circular balancing number if

$$3+4+\cdots+(x-1)+1=(x+1)+\cdots+m+1,$$

holds for some m. This gives us $m^2+m=2x^2-6$. Thus, a natural number $x>2$ is almost 2-circular balancing number if and only if $8x^2-23$ is a perfect square, say some y^2. This means that to characterize this class of numbers, we have to solve the following generalized Pell's equation

$$y^2-8x^2=-23. \tag{34}$$

As already noted in [4], the fundamental solution of the Pell's equation $y^2-8x^2=1$ is $3+\sqrt{8}$, and a fundamental solution of equation (34) is $3+2\sqrt{8}$. So, one class of almost 2-circular balancing numbers can be found from

$$y_n+\sqrt{8}x_n=(3+2\sqrt{8})(3+\sqrt{8})^{n-1},$$

for $n=1,2,\ldots$. Hence, the nth member in this class is given by

$$x_n=\frac{(3+4\sqrt{2})(3+2\sqrt{2})^{n-1}-(3-4\sqrt{2})(3-2\sqrt{2})^{n-1}}{4\sqrt{2}},$$

for $n=1,2,\ldots$. This can be expressed as $x_n=2B_n-3B_{n-1}$, where we use the Binet form for balancing numbers from [1], for $n=1,2,\ldots$.

It is known that $y_{-n}+\sqrt{8}x_{-n}$ is also a solution of equation (34), and using $B_{-n}=-B_n$ we can conclude that $x'_n=3B_{n+1}-2B_n$ is another class of solutions for almost 2-circular balancing numbers. It can be seen that (34) has only two families of solutions, so these are the full list of such almost 2-circular balancing numbers. We can in fact recursively calculate these solutions using the recurrence relations

$$x_{n+1}=6x_n-x_{n-1}$$

and

$$x'_{n+1}=6x'_n-x'_{n-1},$$

with initial values $x_0=3, x_1=2, x'_0=2, x'_1=3$. Thus, we have proved the following theorem.

Theorem 3.1. *The almost 2-circular balancing numbers are the solutions in x of the generalized Pell's equation $y^2-8x^2=-23$ and are partitioned into two classes given by $x_n=2B_n-3B_{n-1}$ and $x'_n=3B_{n+1}-2B_n$ for $n=1,2,\ldots$. They satisfy the recurrence relations $x_{n+1}=6x_n-x_{n-1}$ and $x'_{n+1}=6x'_n-x'_{n-1}$, with initial values $x_0=3, x_1=2, x'_0=2, x'_1=3$.*

4 Almost 3 Circular Balancing Numbers

By the definition, a natural number x is an almost 3-circular balancing number if

$$4+5+\cdots+(x-1)+1=(x+1)+\cdots+m+1+2,$$

holds for some m. This gives us $m^2 + m = 2x^2 - 16$. Thus, a natural number $x > 2$ is almost 3-circular balancing number if and only if $8x^2 - 63$ is a perfect square, say some y^2. This means that to characterize this class of numbers, we have to solve the following generalized Pell's equation

$$y^2 - 8x^2 = -63. \tag{35}$$

A fundamental solution of equation (35) is $3 + 3\sqrt{8}$. So, one class of almost 3-circular balancing numbers can be found from

$$y_n + \sqrt{8}x_n = (3 + 3\sqrt{8})(3 + \sqrt{8})^{n-1},$$

for $n = 1, 2, \ldots$. Hence, the nth member in this class is given by

$$x_n = \frac{(3 + 6\sqrt{2})(3 + 2\sqrt{2})^{n-1} - (3 - 6\sqrt{2})(3 - 2\sqrt{2})^{n-1}}{4\sqrt{2}},$$

for $n = 1, 2, \ldots$. This can be expressed as $x_n = 3B_n - 6B_{n-1}$, where we use the Binet form for balancing numbers from [1], for $n = 1, 2, \ldots$.

Further, we can conclude that $x'_n = 6B_{n+1} - 3B_n$ is another class of solutions for almost 3-circular balancing numbers. It can be seen that (35) has only two families of solutions, so these are the full list of such almost 3-circular balancing numbers. We can in fact recursively calculate these solutions using the recurrence relations

$$x_{n+1} = 6x_n - x_{n-1}$$

and

$$x'_{n+1} = 6x'_n - x'_{n-1},$$

with initial values $x_0 = 3, x_1 = 6, x'_0 = 6, x'_1 = 3$. Thus, we have proved the following theorem.

Theorem 4.1. *The almost* 2*-circular balancing numbers are the solutions in x of the generalized Pell's equation $y^2 - 8x^2 = -63$ and are partitioned into two classes given by $x_n = 3B_n - 6B_{n-1}$ and $x'_n = 6B_{n+1} - 3B_n$ for $n = 1, 2, \ldots$. They satisfy the recurrence relations $x_{n+1} = 6x_n - x_{n-1}$ and $x'_{n+1} = 6x'_n - x'_{n-1}$, with initial values $x_0 = 3, x_1 = 6, x'_0 = 6, x'_1 = 3$.*

5 Almost 4 Circular Balancing Numbers

By the definition, a natural number x is an almost 4-circular balancing number if

$$5 + 6 + \cdots + (x - 1) + 1 = (x + 1) + \cdots + m + 1 + 2 + 3,$$

holds for some m. This gives us $m^2 + m = 2x^2 - 30$. Thus, a natural number $x > 2$ is almost 4-circular balancing number if and only if $8x^2 - 119$ is a perfect square, say some y^2. This means that to characterize this class of numbers, we have to solve the following generalized Pell's equation

$$y^2 - 8x^2 = -119. \tag{36}$$

A fundamental solution of equation (36) is $3 + 4\sqrt{8}$. So, one class of almost 4-circular balancing numbers can be found from

$$y_n + \sqrt{8}x_n = (3 + 4\sqrt{8})(3 + \sqrt{8})^{n-1},$$

for $n = 1, 2, \ldots$. Hence, the nth member in this class is given by

$$x_n = \frac{(3+8\sqrt{2})(3+2\sqrt{2})^{n-1} - (3-8\sqrt{2})(3-2\sqrt{2})^{n-1}}{4\sqrt{2}},$$

for $n = 1, 2, \ldots$. This can be expressed as $x_n = 4B_n - 9B_{n-1}$, where we use the Binet form for balancing numbers from [1], for $n = 1, 2, \ldots$.

Further, we can conclude that $x'_n = 9B_{n+1} - 4B_n$ is another class of solutions for almost 4-circular balancing numbers.

It can be seen that (36) has four families of solutions. Another fundamental solution of equation (36) is $9 + 5\sqrt{8}$. So, one class of almost 4-circular balancing numbers can be found from

$$\overline{y}_n + \sqrt{8}\overline{x}_n = (9+5\sqrt{8})(3+\sqrt{8})^{n-1},$$

for $n = 1, 2, \ldots$. Hence, the nth member in this class is given by

$$\overline{x}_n = \frac{(9+10\sqrt{2})(3+2\sqrt{2})^{n-1} - (9-10\sqrt{2})(3-2\sqrt{2})^{n-1}}{4\sqrt{2}},$$

for $n = 1, 2, \ldots$. This can be expressed as $\overline{x}_n = 15B_n - 6B_{n-1}$. The other family of solutions is given by $\overline{x}'_n = 6B_{n+1} - 15B_n$.

We can again recursively calculate these solutions using the recurrence relations

$$x_{n+1} = 6x_n - x_{n-1}, x'_{n+1} = 6x'_n - x'_{n-1},$$

and

$$\overline{x}_{n+1} = 6\overline{x}_n - \overline{x}_{n-1}, \overline{x}'_{n+1} = 6\overline{x}'_n - \overline{x}'_{n-1},$$

with initial values $x_0 = 4, x_1 = 9, x'_0 = 9, x'_1 = 4, \overline{x}_0 = 6, x_1 = 15, x'_0 = 15, x'_1 = 6$. Thus, we have proved the following theorem.

Theorem 5.1. *The almost* 4*-circular balancing numbers are the solutions in* x *of the generalized Pell's equation* $y^2 - 8x^2 = -119$ *and are partitioned into four classes given by* $x_n = 4B_n - 9B_{n-1}$, $x'_n = 9B_{n+1} - 4B_n$, $\overline{x}_n = 15B_n - 6B_{n-1}$ *and* $\overline{x}'_n = 6B_{n+1} - 15B_n$ *for* $n = 1, 2, \ldots$. *They satisfy the recurrence relations* $x_{n+1} = 6x_n - x_{n-1}$, $x'_{n+1} = 6x'_n - x'_{n-1}$, $\overline{x}_{n+1} = 6\overline{x}_n - \overline{x}_{n-1}$, $\overline{x}'_{n+1} = 6\overline{x}'_n - \overline{x}'_{n-1}$,*, with initial values with initial values* $x_0 = 4, x_1 = 9, x'_0 = 9, x'_1 = 4, \overline{x}_0 = 6, x_1 = 15, x'_0 = 15, x'_1 = 6$.

6 Almost k Circular Balancing Numbers

By the definition, a natural number x is an almost k-circular balancing number if and only if $8x^2 - 8k^2 + 9$ is a perfect square, say some y^2. This means that to characterize this class of numbers, we have to solve the following generalized Pell's equation

$$y^2 - 8x^2 = -8k^2 + 9 \tag{37}$$

A fundamental solution of equation (37) is $3 + k\sqrt{8}$. So, one class of almost k-circular balancing numbers can be found from

$$y_n + \sqrt{8}x_n = (3+k\sqrt{8})(3+\sqrt{8})^{n-1},$$

for $n = 1, 2, \ldots$. Hence, the nth member in this class is given by

$$x_n = \frac{(3+2k\sqrt{2})(3+2\sqrt{2})^{n-1} - (3-2k\sqrt{2})(3-2\sqrt{2})^{n-1}}{4\sqrt{2}},$$

for $n = 1, 2, \ldots$. This can be expressed as $x_n = kB_n - (3k-3)B_{n-1}$, where we use the Binet form for balancing numbers from [1], for $n = 1, 2, \ldots$.

Further, we can conclude that $x'_n = (3k-3)B_{n+1} - kB_n$ is another class of solutions for almost k-circular balancing numbers. We can in fact recursively calculate these solution classes using the recurrence relations

$$x_{n+1} = 6x_n - x_{n-1}$$

and

$$x'_{n+1} = 6x'_n - x'_{n-1},$$

with initial values $x_0 = 3k-3, x_1 = k, x'_0 = k, x'_1 = 3k-3$. Thus, we have proved the following theorem.

Theorem 6.1. *The almost k-circular balancing numbers are the solutions in x of the generalized Pell's equation $y^2 - 8x^2 = -8k^2 + 9$ and we can always get at least two classes of solution families, given by $x_n = kB_n - (3k-3)B_{n-1}$ and $x'_n = (3k-3)B_{n+1} - kB_n$ for $n = 1, 2, \ldots$. They satisfy the recurrence relations $x_{n+1} = 6x_n - x_{n-1}$ and $x'_{n+1} = 6x'_n - x'_{n-1}$, with initial values $x_0 = 3k-3, x_1 = k, x'_0 = k, x'_1 = 3k-3$.*

7 d-sub Circular Balancing Numbers

We can actually extend our Definition 1.3 into a more general class of numbers, as below.

Definition 7.1. *Let k and d be fixed positive integers. We call a positive integer n, a d-sub k-circular balancing number if the Diophantine equation*

$$(k+1) + (k+2) + \cdots + (n-1) + d = (n+1) + (n+2) + \cdots + m + (1 + 2 + \cdots + k - 1)$$

holds for some natural number m. If $d < 0$, then we call such numbers d-super circular balancing numbers.

Clearly, if $d = 1$, then we have the almost circular balancing numbers. This type of numbers might be worth a self-study, moreover they can give further results like the following representative ones.

Theorem 7.2. *If $D = k^2 + 1$, then the D-subbalancing numbers are the d-sub k circular balancing numbers for $d = -1$.*

Theorem 7.3. *The -2-sub 2-circular balancing numbers are the solutions in x of the generalized Pell's equation $y^2 - 8x^2 = -47$ and are partitioned into two classes given by $x_n = 9B_n - 4B_{n-1}$ and $x'_n = 4B_{n+1} - 9B_n$ for $n = 1, 2, \ldots$. They satisfy the recurrence relations $x_{n+1} = 6x_n - x_{n-1}$ and $x'_{n+1} = 6x'_n - x'_{n-1}$, with initial values $x_0 = 4, x_1 = 9, x'_0 = 9, x'_1 = 4$.*

We leave the proofs of the above two results to the reader. We also believe that the following result is true, which we state as a conjecture.

Conjecture 7.4. *There are no D-subbalancing numbers, if $D = 4k^2 + 1$ for some natural number k. In other words, there are no -1-sub k-circular balancing numbers if k is even.*

We close this paper with the remark that, several types of generalized perfect numbers have been studied by various authors (see [3] and the references within). It might be interesting to adapt some of these type of numbers into the balancing numbers setting.

8 Bibliography

[1] A. Behera and G. K. Panda. On the square roots of triangular numbers. *Fibonacci Quart.*, 37(2):98–105, 1999.

[2] Ravi Kumar Davala and G. K. Panda. Subbalancing numbers. *Matematika (Johor)*, 34(1):163–172, 2018.

[3] Alexandre Laugier, Manjil P. Saikia, and Upam Sarmah. Some results on generalized multiplicative perfect numbers. *Ann. Univ. Ferrara Sez. VII Sci. Mat.*, 62(2):293–312, 2016.

[4] A. K. Panda and G. K. Panda. Circular balancing numbers. *Fibonacci Quart.*, 55(4):309–314, 2017.

[5] G. K. Panda and Ravi Kumar Davala. Perfect balancing numbers. *Fibonacci Quart.*, 53(3):261–264, 2015.

[6] G. K. Panda and A. K. Panda. Almost balancing numbers. *J. Indian Math. Soc. (N.S.)*, 82(3-4):147–156, 2015.

[7] G. K. Panda and S. S. Rout. Gap balancing numbers. *Fibonacci Quart.*, 51(3):239–248, 2013.

[8] S. S. Rout and G. K. Panda. k-gap balancing numbers. *Period. Math. Hungar.*, 70(1):109–121, 2015.

A Survey on Supermodular Games

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Abstract. In this article we will learn in short about game theory with special reference to cooperative games. It will be followed by an introduction to a cooperative game called supermodular games. By making use of the solution concept called core of a game, we will go through some important results on supermodularity. We will wind up the article with an introduction to decomposable games followed by its necessary and sufficient condition for supermodularity.

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1 INTRODUCTION

Game theory can be defined as the study of mathematical models of strategic interactions which may include cooperation and conflict between the rational decision makers [CMS15].It provides mathematical techniques for examining situations where two or more than two individuals/players make decisions for each other's benefit. In other words game theory analyses situations where the decisions of the participating players make impact on each other's payoff or interests which may be oblivious to the participants in some cases. As mentioned, game theory deals with situations where rational decision makers interact and try to get fruitful outcomes from own's point of view, so Aumann [Aum87] suggested "Interactive Decision Theory" as an alternative term to game theory. In game theory we call the participating individuals/groups as players. We basically make two assumptions about the players, rational and intelligent [Mye97]. Rationality here means that the player is aware with the rules of the game along with the outcomes. Game theory is widely used in day to day life. Major applications of game theory is in the field of economics, political science, diplomacy, computer science, biology, psychology etc. Remarkable research works have been carried out throughout the world in the fields mentioned above using the concept of game theory by which one can expect optimal results.

Use of game theory in ancient time can be conceived from the problem of three widows, which is included in the Babylonian Talmud [Wal12], the foundation book of Jewish religion as well as

its civil and criminal law during $0 - 500$ AD. A theorem on game theory which is considered to be the first formal work in this field was developed in 1913 by Zermelo, popularly known as Zermelo's Theorem [Zer13]. The theorem asserted that in the game of chess either white has a winning strategy or black has or each of them can always enforce a draw. Modern game theory began with the two-person zero-sum game developed by von Neumann [vN28] in 1928, where loss of one person is gain for the other and vice-versa, further it was followed by a seminal book "The Theory of Games and Economic Behaviour" authored by von Neumann and Morgenstern [vN28] in 1944, which is regarded as the pioneer book in the field of game theory. This book persuaded many young researchers and academicians to work extensively in the field of game theory and its applications. One researcher who contributed significantly in the field of game theory was John Nash, who in the early of 1950s initiated the game theoretic approach of study of bargaining along with the popular equilibrium called "Nash Equilibrium" [Nas50]. In 1950, Melin Dresher and Merill Flood carried out a work under the aegis of Rand Corporation leading to milestone problem known as Prisoner's Dilemma [Pet08]. L Shapley in 1953 gave a solution concept in a cooperative game with n players, known as Shapley Value [Sha53], further Gillies [Gil59] in 1959 suggested the core of cooperative games as a general solution concept.
Based on the conditions and rules of dealings, we consider three major classes in game theory [Wal12].

- Games in extensive form (tree games):
- Games in strategic form.
- Games in coalitional form.

The first two classes of games belong to **non-cooperative** games and the third class belongs to **cooperative** games. Non-cooperative games are those where action of each individual player is primitive. It is not fair to say that non-cooperative game is applicable only in the situation of conflicts, in fact it is just that each individual player and the preferences of the player provide the basic modelling unit. Some well known examples of non-cooperative games are [Pet08]-

- The battle of the Brismarck Sea.
- Matching Pennies.
- Prisoner's dilemma.
- Battle of Sexes.

Cooperative games are based on cooperation or coalition among the players of the game unlike in non-cooperative games, where there is no cooperation or alliances between the players of the game. Thus in cooperative we observe competition among the coalitions in the set of players rather than competition among the players. Cooperative games are further categorized into two parts, games with **transferable utility (TU**-games) and games with **non-transferable utilities(NTU-**games). In transferable utility game, we assign a value to each possible coalition within the game whereas in the other type, the opportunities available at the disposal of a coalition is represented by a set of utility vectors instead of a single number/formula. Once the value of a coalition is obtained in a TU-game, a natural question arises as how to distribute the profit amongst the players such that the distribution is fair and rational as per each player's contribution in the game. For this, we intuitively go for a solution concept, specially the **core**, which consists of payoff vectors satisfying some conditions discussed later in the study. In our study here, focus will be on a special type of TU-game called as the **Supermodular/convex** games. Concept of supermodular games was introduced by Shapley [Sha71] in 1971, where the inceptive for joining a bigger coalition for any player was maximised. The work on supermodularity of a game was further carried out keeping in mind about the solution concept by Topkis [Top78] in 1978.

2 Preliminaries

Before learning about supermodular games, we will go through some basic definitions and results to be used in the later part of our study. Throughout the study set of all n players will be denoted by $N = \{1, 2, ..., n\}$.

Definition 2.1. *[CMS15] A cooperative game in characteristic function form is an ordered pair (N, v), where $N = \{1, 2, ..., n\}$ is the set of all players and function $v : 2^N \to \boldsymbol{R}$ is the characteristic function which assigns each subset (coalition) of N a real value.*

For each coalition $S \subseteq N$, the value $v(S)$ is called worth of the coalition or coalitional value for S.

Definition 2.2. *[CMS15] In a cooperative game (N, v), for any subset $M \subseteq N$, the game (M, v) involving only the players of coalition M with respect to the same characteristic function v is called as **subgame** of (N, v).*

Definition 2.3. *[CMS15] A cooperative game (N, v) is said to be **super-additive** if $v(S)+v(T) \leq v(S \cup T)$ for all coalitions S and T of N such that $S \cap T = \phi$.*

Definition 2.4. *[CMS15] A cooperative game (N, v) is said to be **sub-additive** if $v(S) + v(T) \geq v(S \cup T)$ for all coalitions S and T of N such that $S \cap T = \phi$.*

Equivalently a cooperative game (N, v) is said super-additive if $(N, -v)$ is sub-additive and vice-versa.

Definition 2.5. *[CMS15] A cooperative game (N, v) is said to be **additive** if $v(S)+v(T) = v(S\cup T)$ for all coalitions S and T of N such that $S \cap T = \phi$.*

Definition 2.6. *[CMS15] In a cooperative game (N, v), **marginal contribution** of player $i \in N$ with respect to a coalition S is the the value $v(S \cup \{i\}) - v(S)$, where $i \in N \setminus S$*

Definition 2.7. *[Pet08] A cooperative game (N, v) is said to be **monotonic** if $v(S) \leq v(T)$ whenever $S \subseteq T \subseteq N$.*

As mentioned in the beginning, if the players in set N with respect to the game (N, v) decide to work together, then a natural question arises as how to distribute the coalitional value among them so that the division is fair enough to everyone. In such cases we go for a solution vector as defined below.

Definition 2.8. *[Pet08] For a cooperative game (N, v), an **allocation vector** or a **payoff vector** is an n-coordinated vector $x = (x_1, x_2, ..., x_n)$,*

x_i in the payoff vector above is the amount received by player i. Further for any coalition $S \subseteq N, x(S)$ is the sum of the payoffs received by players of the coalition S. That is $x(S) = \sum_{i \in S} x_i$.

Definition 2.9. *[Pet08] In a cooperative game (N, v), a payoff vector x is said to be* ***individually rational*** *if $x_i \geq v(\{i\})$.*

Definition 2.10. *[CMS15] In a cooperative game (N, v), a payoff vector x is said to be* ***collective rational*** *if $x(S) \geq \sum_{i \in S} x_i$ for all $S \subseteq N$.*

Definition 2.11. *[CMS15] In a cooperative game (N, v), a payoff vector x is said to be* ***totally rational*** *or* ***pareto efficient*** *if $x(S) = v(N)$.*

In a cooperative game (N, v), **pre-imputation** [CMS15] set is the collection of all pareto payoff vectors, further **imputation** is the set of all pareto and totally rational vectors. Now we are in a position to introduce the concept of core which will be an important tool to determine the supermodularity of a game.

Definition 2.12. *[CMS15] In a cooperative game (N, v), a payoff vector x is said to be in the* ***core****, if x is totally and collective rational. Further the collection of all such vectors is called core of the game and is denoted by $c(v)$. That is $c(v) = \{x \in \boldsymbol{R}^n : x(S) \geq v(S), x(N) = v(N); S \subseteq N\}$*

[CMS15] The Shapley value is an interesting solution concept in a cooperative game. Choosing a particular solution concepts becomes an ambiguous work as it may not seem reasonable to many players. Shapley [Sha53] characterised a unique solution using a collection of intuitively reasonable axioms.

Definition 2.13. *[CMS15] In a cooperative game (N, v), the* ***Shapley Value*** *ϕ is the solution $(\phi_1^{sh}, \phi_2^{sh}, ..., \phi_n^{sh})$, where $\phi_i^{sh} = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(|N|-|S|-1)!}{|N|!}[v(S \cup \{i\}) - v(S)]$ for all coalitions S in N.*

Another way to calculate the Shapley value is by using the permutations of the player set N as average of the marginal vectors of the game. That is $\phi_i^{sh} = \frac{1}{|N|!} \sum_{\pi \in \Pi} m^\pi$, where Π is the collection of all permutations of N and m^π is the marginal contribution of player$i$ with respect to all coalitions. It is to be noted here that the Shapley value is unique.

3 Supermodular Games

In this section we will study about supermodular games followed by some important results based on the supermodularity of a cooperative game using greedy algorithm. Further we will study decomposable games and the criteria for its supermodularity.

Definition 3.1. *[Top98] A Cooperative game with transferable utility or simply TU game (N, v) on the set of players $N = \{1, 2, ..., n\}$ is said to be supermodular or convex if $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$ for all subsets S and T of N.*

Before discussing various examples and results based on the game defined above we will prove an important result which allows us to prove the supermodularity of a game in more than a way.

Theorem 3.2. *[Sha71] A game (N, v) is supermodular if and only if $v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T))$ for all subsets S and T of N such that $S \subseteq T \subseteq N$ and $i \epsilon N \setminus T$.*

Remark: From now onwards, we can use any one of the following ways to establish the supermodularity of a cooperative game.

$$v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$$

$$v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T))$$

$$v(S \cup \{i\}) - v(S) \leq v(S \cup \{i, j\}) - v(S \cup \{j\}),$$

where $S, T \subseteq N$ and i,j are players.

Supermodularity of a game tries to persuade its players for a bigger coalition as from the definition itself it is clear that for higher coalitions the outcome is maximised. There are many practical examples of supermodularity out of which some popular examples are:

- The Airport Profit Game [Top98].
- The Bankruptcy Game [CMT87].
- The Monopoly Firm Game [Top98].

For a permutation π of set N, consider the set $S(\pi, j) = \{\pi(1), \pi(2), ..., \pi(j)\}$ for $j = 1, 2, ..., n$. Let Π be the collection of all permutations of the set N, then for $\pi \in \Pi$, $x(\pi)$ is the payoff vector generated by the **greedy algorithm** [Top98] with permutation π defined as $x(\pi)_{\pi(i)} = v(S(\pi, i)) - v(S(\pi, i - 1))$ for all $i \in N$. In simple words, for a given permutation π, greedy algorithm with permuatation π gives the marginal contribution of player $\pi(i)$ with respect to the coalition $S(\pi, j) = \{\pi(1), \pi(2), ..., \pi(j)\}$.
Theorem below shows that the the payoff generated by the greedy algorithm is in the core in a supermodular game also the Shapley value.

Theorem 3.3. *[Sha53] Suppose (N, v) is a supermodular game.*
(a) For each permutation π of the set of players N, the payoff vector $y(\pi)$ generated by the greedy algorithm is in the core.
(b) The core is non empty.
(c) The shapley value is in the core.

Lemma discussed below shows us a result where the sum of the payoffs generated by greedy algorithm is equal to the value of the game in a coalition where the elements are arranged in the order $\pi(n), \pi(n-1), ..., \pi(1)$.

Lemma 3.4. *[Sha53] Suppose that (N, v) is a supermodular game, π is any permutation of players N, and $x(\pi)$ is the payoff vector generated by the greedy algorithm with the permutation π. Then*
(a)$\sum_{i \in S(\pi,j)} x(\pi)_i = v(S(\pi, j))$ for $j = 1, 2, ..., n$.
(b)$x(\pi)$ simultaneously maximises $\sum_{i \in N \setminus S(\pi,j)} x(\pi)_i$ for $j = 1, 2, ..., n$ over all payoff vectors x in the core.
(c) $x(\pi)$ lexicographically maximises $(x_{\pi(n)}, x_{\pi(n-1)}, ..., x_{\pi(1)})$ over all payoff vectors x in the core.

As of now, we studied the payoff vectors generated by greedy algorithm using permutation of the set of players. Lemma below gives the necessary and sufficient condition for the uniqueness of the payoff vectors generated by the greedy algorithm.

Lemma 3.5. *[Sha53] For a supermodular game (N, v),the characteristic function $v(S)$ is strictly supermodular on $P(N)$ if and only if the $n!$ payoff vectors generated by the greedy algorithm with the $n!$ different permutations of the players of N are distinct.*

Theorem below shows the converse of Theorem 3.3, that if every payoff vector generated by the greedy algorithm wit different permutations of N is in the core, then the game is supermodular. In other words we have another criteria to prove the supermodularity of a game provided the payoff vectors generated by the greedy algorithm is in he core.

Theorem 3.6. *[Ich81] If (N, v) is a cooperative game and the payoff vector generated by the greedy algorithm for each permutation of the payers is in the core, then the cooperative game (N, v) is a supermodular game.*

Let us recall that **extreme point** in a set is a point which cannot be represented as convex combination of any two other points of the set. Theorem below gives us a condition when a payoff vector becomes an extreme point of the core.

Theorem 3.7. *Consider a supermodular game.*
(a)[Sha53] A payoff vector is an extreme point of the core if and only if it is generated by the greedy algorithm with some permutation of the set of players.
(b)[Top98] A payoff vector is an extreme point of the core if and only if it is a convex combination of (at most n+1) payoff vectors generated by the greedy algorithm with some permutation of the set of players.

4 Supermodularity of Decomposable Games

Here we are going to introduce another type of cooperative game called decomposable and will analyse the relation between decomposable and supermodular game.

Let $\{N_1, N_2, N_3, ..., N_p\}$ be a partition of the set of players N such that $p \geq 2$. Then a game (N, v) is said to be **decomposable** [Sha71] with respect to the partition P, if v is additive across the partition P. That is $v(S) = v(S \cap N_1) + v(S \cap N_2) + ... + v(S \cap N_p)$, for all $S \subseteq N$. The game above is completely determined by its value on the subsets N_i of N is called as **components** of the decomposition.

Theorem below proves the necessary and sufficient condition for a decomposable game to be supermodular.

Theorem 4.1. *[Sha71] (a)A decomposable game is supermodular if and only if each component is supermodular.*
(b) A supermodular game is decomposable if and only if $v(N) = v(N_1) + v(N_2) + v(N_3) + ... + v(N_p)$ holds for some partition $\{N_1, N_2, N_3, ..., N_p\}$ of N into $p \geq 2$ nonempty subsets.

Corollary 4.2. *[Sha71] A strictly supermodular game is indecomposable.*

Conclusion: In this article we have discussed about the supermodularity of a cooperative game. We learned about the greedy algorithm with respect to a given permutation of the set of players. Further with the use of the greedy algorithm, we learned to analyse the core of a supermodular game and tried to figure out the game in the other way around. Finally we ended up with an introduction to decomposable games followed by its necessary and sufficient condition for supermodularity. The topic is interesting from both theoretical and practical point of view. Using the notion of supermodularity, many games can be further analysed and suitable results can be obtained for an easy access to the solution concepts like core, the Shapley value etc.

5 Bibliography

[Aum87] R. J. Aumann. *In The New Palgrave Dictionary of Economics.* Palgrave Macmillan, 1987.

[CMS15] S. R. Chakravarty, M. Mitra, and P. Sarkar. *A Course on Cooperative Game Theory.* Cambridge University Press, New Delhi, 2015.

[CMT87] I. J. Curiel, M. Mashcler, and S. Tijs. Bankruptcy games. *Zeitschrift fur Operations Research*, 31:143–159, 1987.

[Gil59] D. B. Gillies. *Solutions to general non-zero-sum games.* Princeton University Press, Princeton, 1959.

[Ich81] T. Ichiishi. Super-modularity: Applications to convex games and to the greedy algorithm for lp. *Mathematische Annalen*, 25:283–286, 1981.

[Mye97] R. Myerson. *Game Theory: Analysis of Conflict.* Harvard University Press, 1997.

[Nas50] J. F. Nash. *Equilibrium points in n-person games.* Proceedings of the National Academy of Science of the United States of America, 1950.

[Pet08] H. Peters. *Game Theory, A Multi-Leveled Approach.* Springer-Verlag Berlin Heidelberg, 2008.

[Sha53] L. Shapley. *A value of n-person games.* Princeton University Press, Princeton, 1953.

[Sha71] L. Shapley. Cores of convex games. *Mathematische Annalen*, 1:11–26, 1971.

[Top78] D. M. Topkis. Minimizing a submodular function on a lattice. *Operations Research*, 26(2):305–321, 1978.

[Top98] D. M. Topkis. *Supermodularity and Complementarity.* Princeton University Press, 1998.

[vN28] J. von Neumann. Zur theorie der gesellschaftsspiele. *Mathematische Annalen*, 100(1):295–320, 1928.

[Wal12] P. Walker. *A Chronology of Game Theory.* ://www:econ:canterbury:ac:nz/personal pages/paul walker/gt/hist.html, 2012.

[Zer13] E. Zermelo. *Uber eine Anwendung der Mengenlehre auf die Theorie des Schachspiels.* Proceedings of the Fifth International Congress of Mathematicians 2, 1913.

A Survey On Ideal Convergence in Topologicical spaces

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Abstract. Here we present a short survey on development of ideal convergence theory around topological spaces and some of its applications. Some recent results are included with proper references and directions of propagation for further advancements.

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1 Introduction

The concept of statistical convergence is introduced by H Fast [6] in 1951 and from then several new type of convergence of sequences appeared, many of them are related to statistical convergence. In 2000 Šalát and Kostyrko introduced I-Convergence in [11] and the concept of I-convergence gives a unifying approach to these type of convergence.
The idea of I-convergence has been extended from real number space to metric space and many other spaces in recent works. Later B. K. Lahiri and P. Das extended this idea to an arbitrary topological space in [1] and observes that the some properties are preserved in a topological space. In 2008, they also introduced the idea of I-convergence of Nets in [2] a topological space.
The recent advancements in ideal convergence theory ask for some preliminaries.

Definition 1.1. *Let I be an ideal defined on $\mathbb{N}$ where ideals are collection of subsets of $\mathbb{N}$ closed under finite unions and subsets and containing atleast the finite subsets of $\mathbb{N}$.*

. Fin and I_0 are two basic ideals defined as follows:

$$\text{Fin} = \text{collection of all finite subsets of } \mathbb{N}.$$
$$I_0 = \text{Subsets of natural number with density } 0.$$
$$A \in I_d \text{ if and only if } limsup_{n\to\infty} \frac{|A \cap \{1,2,\ldots,n\}|}{n} = 0.$$

If we consider an ideal I in P($\mathbb{N}$), two additional subsets of P($\mathbb{N}$) namely I^*, I^+ occurs.
We denote $I^* = \{A \subset \mathbb{N} : A^c \in I\}$, the filter dual of I and I^+= collection of all subsets which doesn't belongs to I.

Remark 1.2. *Clearly,* $I^* \subset I^+$*.*

Definition 1.3. *[12] Let X be a topological space. Then, a sequence $x = (x_n)$ is said to be I-convergent to ξ, shortened with $x_n \to_I \xi$, whenever $\{n : x_n \in U\} \in I$ for all neighborhoods U of ξ.*

Remark 1.4. *i.e (X, T, I) doesn't corresponds to $(X,\ T^{'},\ Fin)$.*

Note 1.5.

Fin- convergence is essentially called general convergence.

If $I = I_0$ then this mode of convergence is called the Statistical convergence. Beginning with Fast[6], Salat[4], Tripathy [13], Kostyrko[12] and many renowned researcher has been pushing this topic into the depth of advancement.

Example 1.6. ***I-convergence doesn't corresponds to topology[10];*** *Assume $I \neq Fin$ and (X, T) is topological space, where $|X| \geq 2$ and T is not trivial topology T_0. Let $l \in \boldsymbol{I} \setminus Fin$. Here I is infinite. Fix distinct $a, b \in X$ and define the sequence (x_n) by $x_n = a$ whenever $n \notin l$ and $x_n = b$ otherwise.*
It follows that $x_n \to_I a$ in (X, T). let us assume, for the sake of contradiction, there exists a topology T' such that $x_n \to a$ in (X, T'). If there is a T'-neighborhood U of a such that $b \notin U$, then $\{n : x_n \notin U\} = I$. This is impossible, since l is not finite. Hence $b \in U$. Hence $b \in U$ whenever $a \in U$. By the arbitrariness of a and b, we conclude that $T' = T_0$. The converse is false: given $U \in T \setminus T_0$ and $u \notin U$, then the constant sequence (u) is not I-convergent to l provided that $l \in U$.

Given a topological space (X, T) and an ideal I, define the family as

$$T(I) := \{F^c \subset X : F = \bigcup_{x \in F'} C_x(I)\}$$

that is, F is $T(I)$-closed if and only if it is the union of I-cluster points of F-valued sequences. In particular, it is immediate that $T = T(Fin)$.

Lemma 1.7. $T \subset T(I)$*.[10]*

Interestingly, The following theorem in the article [10] by Leoneti, ensures the equality with certain condition on X.

Theorem 1.8. *Assume that one of the following conditions holds:*

- *X is sequentially strongly Lindelof and I is a P-ideal;*
- *X is first countable.*

Then $T = T(I)$.

Statistical limit points and statistical cluster points (i.e., I_0-limit points and I_0-cluster points, resp.) of real sequences were introduced by Fridy as in [12]. Similarly I-cluster point and I-limit point set is defined in [4, 12].

Definition 1.9. *let $C_x(I)$ denote the set of I-cluster points of x, that is, the set of all $\xi \in X$ such that $\{n : x_n \in U\} \in I^+$ for all neighborhoods U of ξ*

Definition 1.10. *$y \in X$ is called an I-limit point of x if there exists a set $M = \{m_1 < m_2 < ...\} \subset \mathbb{N}$ such that $M \notin I$ and $lim_{k\to\infty} x_{m_k} = y$.*

Many other results has been established in this decade from statistical background to Ideal theory which can be found in [5, 7, 9]

Lemma 1.11. *[10] Let x and y be sequences taking values in a topological space X and fix ideals $I \subset J$. Then:*

- *$L_x(J) \subset L_x(I)$ and $C_x(J) \subset C_x(I)$;*
- *$L_x(Fin) = C_x(Fin)$, provided X is first countable;*
- *$L_x(I) \subset C_x(I)$;*
- *$C_x(I)$ is closed;*
- *$L_x(I) = L_y(I)$ and $C_x(I) = C_y(I)$ provided $x =_I y$;*
- *$C_x(I) \cap K \neq \phi$, provided $K \subset X$ is compact and $\{n : x_n \in K\} \in I^+$;*
- *$L_x(I) = C_x(I) = \{l\}$ provided $x_n \to_{I'} l$ and X is Hausdorff.*

.

I- limit point and I-cluster point seems related by their very definition. $C_x(I)$ is a closed set in X and It is characterized in [4] as follows:

Theorem 1.12. *Let I be an admissible ideal.*
(i) The set $C_x(I)$is closed in X for each sequence $x = \{x_n\}$ of elements of X.
(ii) Suppose that (X, T) is a separable metric space. Suppose that there exists a disjoint sequence of sets $\{M_n\}$ such that $M_n \in N$ and $M_n \notin I$ for $n \in N$. Then for each closed set $F \in X$ there exists a sequence $x = \{x_n\}$ of elements of X such that $F = C_x(I)$.

Now, we are getting a characterization of $C_x(I)$ with closed sets in X under some suitable conditions in The space X and I.Similar Results for $L_x(I)$ can be expected. Results came in favour of F_σ sets in X. For this, some basics need to be revisited as follows :

Definition 1.13. *[7] An ideal I is said to be a P -ideal (or said to satisfy condition (AP)) if for every sequence $(A_n)_n \in \mathbb{N}$ of elements of I there exists $A_\infty \in I$ such that $A_n \setminus A_\infty$ is a finite set for every $n \in N$.*

After identifying the power set $P(\mathbb{N})$ of N with the Cantor space $C = \{0, 1\}^{\mathbb{N}}$ in a standard manner we may consider an ideal as a subset of C. In particular, an ideal I is analytic if it is a continuous image of a Borel subset of a Polish space.

Definition 1.14. *[7] Let S be a set.*
We say that a map $\Psi : P(S) \to [0, \infty]$ is a sub measure on S if it satisfies the following conditions:

- *$\Psi(\Phi) = 0$ and $\psi(s) < \infty$ for every $s \in S$,*

- *Ψ is monotone: if $A \subset B \subset S$, then $\Psi(A) \subset \psi(B)$,*

- *Ψ is subadditive: if $A, B \subset S$, then $\psi(A \cup B) \leq \Psi(A) + \Psi(B)$.*

A sub measure Ψ on $\mathbb{N}$ is lower semi continuous if for every $A \subset \mathbb{N}$ we have

$$\Psi(A) = lim_{n \to \infty} \Psi(A \cap [1, n]).$$

In article by Das[7], Characterization of F_σ set with $L_x(I)$ was established. Here we have the result:

Theorem 1.15. *Let X be a first countable space. For any sequence $(x_n)_{n \in \mathbb{N}}$ in X the set $L_x(I)$ is an F_σ -set provided I is an analytic P -ideal.*

Theorem 1.16. *[7] Let X be a space with hcld(X) $= \omega$. Then for each F_σ -set A in X there exists a sequence $x = (x_n)_{n \in \mathbb{N}}$ in X such that $A = L_x(I)$ provided I is an analytic P -ideal.*

According to the article [7], an ideal I is F_σ if and only if there exists a lower semi continuous sub measure Ψ such that $I = \{A \subset N : \Psi(A) < \infty\}$, with $\Psi(N) = \infty$.

Theorem 1.17. *[14] Let $x = (x_n)$ be a sequence taking values in a first countable space X and let I be an F_σ-ideal. Then $L_x(I) = C_x(I)$. In particular, $L_x(I)$ is closed.*

Combine scenario for analytic P-ideals, the property that the set of I-limit points is always closed, characterizes the subclass of F_σ-ideals:

Theorem 1.18. *[14] Let X be a first countable space which has a non-isolated point. Let also I_Ψ be an analytic P-ideal. Then the following are equivalent:*

- *I_σ is also an F_σ-ideal;*
- *$L_x(I) = C_x(I_\Psi)$ for all sequences x;*
- *$L_x(I_\Psi)$ is closed for all sequences x;*
- *there does not exist a partition $\{A_n : n \in N\}$ of N such that $||A_n||_\Psi > 0$ for all n and $lim \bigcup_{k>n} ||A_k||_\Psi = 0$.*

Above theorem can be found in article [ideal relationship].
As defined a topological space X is said to be locally compact if for every $x \in X$ there exists a neighborhood U of x such that its closure U is compact,

Theorem 1.19. *Let $x = (x_n)$ be a sequence taking values in a locally compact first countable space and fix an analytic P-ideal I_Ψ. Then each isolated I_Ψ-cluster point is also an I_Ψ-limit point.*

Definition 1.20. *A set S is discrete in a larger topological space X if every point x in S has a neighborhood U such that $S \cap U = \{x\}$.*

following results is a direct implication of above theorem:

Corollary 1.21. *Let x be a real sequence for which $C_x(I)$ is a discrete set. Then $L_x(I) = C_x(I)$.*

Following theorem is a kind of converse of **theorem 1.17** can be found in recent publication by Leonetti[14].

Theorem 1.22. *Let X be a separable metric space and fix sets $A \subset B \subset C \subset X$ such that A is an F_σ-set and B, C are closed sets such that the set S of isolated points of B is contained in A and $F := B \setminus S$. is non-empty. Moreover, assume that there exists an atomless strictly positive Borel probability measure $\mu_F : B(F) \to [0,1]$. Then there exists a sequence x taking values in X such that $L_x(I) = A$, $C_x(I) = B$, and $L_x(Fin) = C$.*

Theorem 1.23. *[14] Let X be a first countable space where all closed sets are separable and let $I \neq Fin$ be an F_σ-ideal. Fix also closed sets B, $C \subset X$ such that $\phi \neq B \subset C$. Then there exists a sequence x such that $L_x(I) = C_x(I) = B$ and $L_x(Fin) = C$.*

It will interesting to consider the topological nature of the set of I-limit points when I is neither F_σ- nor analytic P-ideal. It is exciting that there exist an ideal I and a real sequence x such that $L_x(I)$ is not an F_σ-set. One can find it in [14] by Leonetti.
Lastly, An open question arise in this context mentioned in [14] by leonetti as follows:

whether there exists a real sequence x and an ideal I such that $L_x(I)$ is not Borel measurable.

2 Bibliography

[1] B.K. Lahiri and P. Das, *I and I^*-convergence in topological spaces*, Math. Bohem., **130** (2005), 153-160.

[2] B.K. Lahiri , Das P., *I and I*-convergence of nets*, Real Analysis Exchange, 33 (2) (2007/2008), 431-442.

[3] D.N. Georgiou, S.D. Iliadis, A.C. Megaritis, G.A. Prinos *Ideal-convergence classes* Topol. Appl., 222 (2017), pp. 217-226.

[4] Kostyrko, P., Salat, T., Wilczynski, W.: *I-convergence*. Real Analysis Exch. 26(2000/2001), 669-685.

[5] M. Balcerzak, Sz. G lab, and A. Wachowicz, *Qualitative properties of ideal convergent subsequences and rearrangements*, Acta Math. Hungar. 150 (2016), no. 2, 312-323.

[6] H. Fast, *Sur la convergence statistique*, Coll. Math., 2 (1951), 241-244.

[7] P. Das, *Some further results on ideal convergence in topological spaces*, Topol. Appl. 159 (10-11) (2012) 2621-2626.

[8] P. Leonetti, *Thinnable ideals and invariance of cluster points*, Rocky Mountain J. Math., **48(6)** (2018), 1951-1961. .

[9] P. Leonetti, *Limit points of subsequences*, preprint, last updated: Dec 31, 2017 (arXiv:1801.00343).

[10] P. Leonetti and F. Maccheroni, *Ideal cluster points in topological spaces*, preprint, last updated:Nov 22, 2017 (arXiv:1707.03281).

[11] P. Kostyrko, T. Salat, W. WilczyÃ…Â„ski *I-convergence*, Real Analysis Exchange, 26 (2) (2000/01), pp. 669-685.

[12] P. Kostyrko, M. Macaj, T. Šalat, *Statistical convergence and I-convergence*, Real Analysis Exch (2000/2001) 231-239.

[13] A, T., Tripathy, B. C., Ziman, M.: *A note on I-convergence field*. To appear in Italian J. Pure Appl. Math.

[14] Paolo Leonetti, M. Balcerzak; On the relationship between Ideal limit points and ideal cluster points, Topology and its application (2019) 178-190.

Shift operators: a review on their reducing and minimal reducing subspaces

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Abstract. Weighted shift operators have been studied extensively in the past few decades. Shifts, including both unilateral and bilateral ones, are a basic tool in operator theory. Shift operators were originally studied as scalar shifts, which in due course of time were generalized into operator shifts. In literature, we find various works that have been carried out on shift operators which proves that these operators are a fertile ground for providing examples and counter examples in various branches of operator theory. Shift operators also find their use in various other branches of study such as graph theory, complex analysis and computer science. In this article, we shall see how the study on shift operators has evolved in due course of time, focusing mainly on the reducing and minimal reducing subspaces of these operators.

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1 Introduction

In the branch of operator theory, shift operators are a class of very widely and extensively studied linear operators on Hilbert spaces. These operators are of fundamental importance in many parts of operator theory. The unilateral shift is not just an isometry, but it is a fundamental building block out of which all isometries are constructed. Some of the adequate and comprehensive references are [Fil], [Nik], [Sar74], [JW07] and [Shi74]. The shift operators have many interesting properties, both analytic and algebraic and even though the properties may not have immediate visible application, still they prove to be very valuable.

We begin with the introduction of the unilateral shift operator on a separable Hilbert space H. Let $\{e_n\}_{n=0}^{\infty}$ be an orthonormal basis of H. Then the operator U defined by

$$Ue_n = e_{n+1} \text{ for } n = 0, 1, 2, \ldots$$

is called the unilateral shift operator on H.
Its adjoint operator is given by

$$U^*e_0 = 0, \text{ and } U^*e_n = e_{n-1} \text{ for } n = 1, 2, 3, \dots$$

and is called the backward unilateral shift operator. The matrix representation of U with respect to the orthonormal basis $\{e_n\}_{n=0}^{\infty}$ is

$$\begin{pmatrix} 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 1 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 1 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 1 & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \\ \cdot & \cdot & \cdot & \cdot & & & \cdot \end{pmatrix}$$

and that of U^* is

$$\begin{pmatrix} 0 & 1 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 1 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 1 & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \\ \cdot & \cdot & \cdot & \cdot & & & \cdot \end{pmatrix}.$$

The unilateral shift operator is injective but not surjective, while its adjoint, the backward unilateral shift is surjective but not injective. Recall that a bounded linear operator T on a Hilbert space H is called an isometry if $\|Tx\| = \|x\|$ for all x in H. Hence, the unilateral shift is an isometry.

The unilateral shift motivated the study of the unilateral weighted shift operators. For a sequence of complex numbers $\{w_n\}_{n=0}^{\infty}$, the unilateral weighted shift operator T is defined on the Hilbert space H as

$$Te_n = w_n e_{n+1} \text{ for } n = 0, 1, 2, \dots$$

The scalars w_n are called the weights of the shift operator T. For better understanding, we further define the unilateral weighted shift on the familiar Hilbert space $\ell^2_+(\mathbb{C})$. Let $\mathbb{C}$ denote the complex plane and $\mathbb{N}_0$ denote the set of non-negative integers. The space $\ell^2_+(\mathbb{C})$ is defined as follows:

$$\ell^2_+(\mathbb{C}) := \{x = (x_0, x_1, \dots) : x_i \in \mathbb{C}, \sum_{i \in \mathbb{N}_0} |x_i|^2 < \infty\}.$$

For a bounded sequence of non-zero scalars $\{\alpha_n\}_{n \in \mathbb{N}_0}$, the unilateral weighted shift T is defined on $\ell^2_+(\mathbb{C})$ as

$$T(x_0, x_1, \dots) = (0, \alpha_0 x_0, \alpha_1 x_1, \dots).$$

Its adjoint T^* is given by

$$T^*(x_0, x_1, \dots) = (\bar{\alpha}_0 x_1, \bar{\alpha}_1 x_2, \dots)$$

and is called the backward unilateral scalar weighted shift with weight sequence $\{\bar{\alpha}_n\}_{n\in\mathbb{N}_0}$.

Similarly, we can define the bilateral scalar weighted shift on the sequence space $\ell^2(\mathbb{C})$, where

$$\ell^2(\mathbb{C}) := \{x = (\dots, x_{-1}, [x_0], x_1, \dots) : x_i \in \mathbb{C}, \sum_{i\in\mathbb{Z}} |x_i|^2 < \infty\}.$$

Then for a sequence of non-zero scalars $\{\alpha_n\}_{n\in\mathbb{Z}}$, the bilateral weighted shift W is defined on $\ell^2(\mathbb{C})$ as

$$W(\dots, x_{-1}, [x_0], x_1, \dots) = (\dots, \alpha_{-2}x_{-2}, [\alpha_{-1}x_{-1}], \alpha_0 x_0, \alpha_1 x_1, \dots).$$

Its adjoint W^* is called the backward bilateral weighted shift. Here, $[\cdot]$ in $x = (\dots, x_{-1}, [x_0], x_1, \dots)$ denotes the central 0th entry of x.

2 Reducing and minimal reducing subspaces

We are mainly concerned about the reducing and minimal reducing subspaces of the shift operators. Hence, we first recall the concerned definitions.

Definition 2.1. *A subspace M of a Hilbert space is called an invariant subspace under an operator T if $T(M) \subseteq M$.*
If a subspace M is invariant under both T and its adjoint T^*, then M is said to be a reducing subspace for T.
A reducing subspace M is said to be a minimal reducing subspace if it does not contain any proper non zero reducing subspace.

We shall now discuss some of the significant results till date in the context of invariant, reducing and minimal reducing subspaces of the shift operators.

In 1949, Beurling [Beu49] explicitly described all the non zero invariant subspaces of the unilateral shift as subspaces of H^2. The space H^2 is the Hilbert space of all analytic functions having power series representations with square summable complex coefficients.

Theorem 2.2. *Beurlings theorem : Every invariant subspace of the unilateral shift other than* 0 *has the form φH^2, where φ is an inner function.*

It must be noted that the unilateral shift operator has in fact no proper reducing subspaces. In an attempt to study more about the properties of these shift operators, the scalar weights were replaced by operator weights and by doing this, a new class of operators called the operator weighted shifts were introduced.

In 1967, N.K. Nikolskii [Nik67] introduced operator weighted shifts as a generalization of scalar weighted shifts. Let K be a separable complex Hilbert space and $\ell^2_+(K)$ be defined as the orthogonal sum of $\aleph_0$ copies of the Hilbert space K with a scalar product defined by

$$\langle f, g\rangle = \sum_{n\in\mathbb{N}_0} \langle f_n, g_n\rangle,$$

for $f = (f_0, f_1, \dots) \in \ell^2_+(K)$, and $g = (g_0, g_1, \dots) \in \ell^2_+(K)$.

Considering $\{A_n\}_{n\in\mathbb{N}_0}$ to be a uniformly bounded sequence of linear operators on K, the operator S on $\ell^2_+(K)$ defined as

$$S(f_0, f_1, \dots) = (0, A_0 f_0, A_1 f_1, \dots)$$

is called an unilateral operator weighted shift with weights $\{A_n\}_{n\in\mathbb{N}_0}$. Clearly, S is bounded and $\|S\| = \sup_n \|A_n\|$.

As in the case of scalar shifts, considering each A_n as the identity operator, unweighted unilateral operator shifts are defined as

$$S_+(f_0, f_1, \dots) = (0, f_0, f_1, \dots).$$

It is an important fact that operator shifts are not just a formal generalization of scalar shifts. For instance, with the help of an operator weighted shift, Pearcy and Petrovic [PP94] proved that an n-normal operator is power bounded if and only if it is similar to a contraction. Since its introduction, operator weighted shifts have been widely studied. For a general understanding of its various properties we refer the following: [Bou06, Gel69, Her90, Jab04, Lam71, LS01, Nik67].

In his paper [Nik67], Nikol'skii considered unilateral operator weighted shifts in the form of S_+R operating on the Hilbert space $\ell^2_+(K)$. Here S_+ is a unilateral unweighted operator shift and R is a multiplication operator. He deduced significant results relating to invariant and reducing subspaces of operator weighted shifts. A few of the important ones are stated below:

Lemma 2.3. *[Nik67] Let T be a bounded operator in the Hilbert space K, and let $T = VR$, where V is an isometric operator and R is a selfadjoint operator. In addition, let the operator R be one-to-one. For the subspace L of H to reduce T, it is necessary and sufficient that L reduces both V and R.*

Using the above lemma, Nikol'skii proved the following important result.

Theorem 2.4. *[Nik67] Let $T = S_+R$ be an operator in $\ell^2_+(K)$. Here, R is a multiplication operator defined as $RX = R(X_0, X_1, \dots) = (R_0X_0, R_1X_1, \dots)$, where the R_i's are self adjoint and one-to-one operators in K for each $i \in \mathbb{N}_0$. Then all reducing subspaces of T are of the form $\hat{L} = \{X = (X_0, X_1, \dots) \in \ell^2_+(K) : X_i \in L, i \in \mathbb{N}_0\}$, where L is a closed subspace of K such that L is invariant under each R_i.*

In 1971, Lambert [Lam71] worked on operator weighted shifts on $\ell^2_+(K)$ with invertible and bounded operator weights. He considered a sequence of invertible bounded linear operators $\{A_n\}_{n=0}^{\infty}$ on K and defined the operator weighted shift S using these weights. He denoted this operator weighted shift as $S \sim \langle A_n \rangle$. In this case, S is called an invertibly weighted shift. He established necessary and sufficient conditions for the unitary equivalence of two such shifts, and also introduced some significant results on reducing subspaces of invertibly weighted shifts. We shall give a brief discussion on his paper.

Let $S \sim \langle A_n \rangle$ be an invertibly weighted shift on the Hilbert space $\ell^2_+(K)$. We construct a sequence $\{S_n\}_{n=0}^{\infty}$ of operators as $S_0 = I$ and $S_n = A_nA_{n-1}\dots A_0$, $n \geq 1$, where I is the identity operator on K. Note that each S_n is an invertible operator on K and $S_{n+1} = A_{n+1}S_n$.

Lemma 2.5. *[Lam71] Let $S \sim \langle A_n \rangle$ be an invertibly weighted shift on the Hilbert space $\ell^2_+(K)$. If M is a reducing subspace of S, then $M = \sum_{n=0}^{\infty} \oplus S_n M_0$ for some subspace M_0 of K.*

In the above lemma, we must see that for any subspace M_0 of K , $M = \sum_{n=0}^{\infty} \oplus S_n M_0$ is an invariant subspace for S, but it is not necessary that it is also reducing. The following result gives us a necessary and sufficient condition for a subspace to be reducing for S.

Theorem 2.6. *[Lam71] Let $S \sim \langle A_n \rangle$ be an invertibly weighted shift on the Hilbert space $\ell^2_+(K)$. Let $M = \sum_{n=0}^{\infty} \oplus S_n M_0$ be a subspace of $l^2_+(K)$. Then the following statements are equivalent:*
(i) M is a reducing subspace of the shift S.
(ii) $S_n M_0$ is invariant for $A^*_{n+1} A_{n+1}$, $n = 0, 1, 2, \ldots$.
(iii) $(S_n M_0)^{\perp} = S_n(M_0^{\perp})$, $n = 0, 1, 2, \ldots$.
(iv) $S^*_n S_n M_0 = M_0$, $n = 0, 1, 2, \ldots$.

Lambert proved an important corollary for this theorem. Let $L(K)$ denote the algebra of all bounded linear operators on K. For an invertibly weighted shift S, let $T(S)$ denote the weakly closed* subalgebra of $L(K)$ that is generated by $\{S^*_n S_n\}_{n=0}^{\infty}$.

Corollary 2.7. *[Lam71] The lattice of the reducing subspaces of the invertibly weighted shift operator S is isomorphic to the lattice of $T(S)$. In particular S is irreducible if and only if $T(S) = L(K)$.*

Later in 1985, Guyker [Guy85] extended this study and established a reducibility criterion for bilateral operator weighted shifts with commuting normal operator weights.

In this line of study, a very interesting work was done by Stessin and Zhu in 2002. In their paper [SZ02], they considered a weighted unilateral shift operator S, with finite multiplicity $N > 1$ on the Hilbert space H^2_w and expressed it in the form of a multiplication operator M_{z^N}.

Let $w = \{w_0, w_1, \ldots\}$ be a sequence of positive numbers. Then the space H^2_w is the Hilbert space of all analytic functions of the form

$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

that are defined in the unit disk $\mathbb{D}$. Also, the norm of these analytic functions are given by

$$\|f\|^2 = \sum_{k=0}^{\infty} w_k |a_k|^2 < \infty.$$

The weights w are taken in such a way that

$$\sup\{\frac{w_{n+1}}{w_n} : n \geq 0\} < \infty.$$

Hence, M_z, the operator of multiplication by z is bounded in H^2_w. Let $N > 1$ be an integer. Then M_{z^N} is the multiplication operator by z^N. Stessin and Zhu considered the weighted unilateral shift operator S on H^2_w as

$$S = M_z^N = M_{z^N}$$

Let $X_n = span\{z^{n+kN} : k = 0, 1, 2, \dots\}$. Clearly, X_n is a reducing subspace for S. The weights w in this paper are classified into two types:
Type I: For each $0 < n, m < N-1$, and $n \neq m$, there exists a some integer $k > 0$ such that

$$\frac{w_{n+kN}}{w_n} \neq \frac{w_{m+kN}}{w_m}.$$

Type II: If w is not of type I.

Base on these types, the important results of this paper are stated below:

Theorem 2.8. *[SZ02] Every reducing subspace X of S in H^2_w contains a minimal reducing subspace. The reducing subspaces X_n, $0 \leq n \leq N-1$, are all minimal. And every minimal reducing subspace of S in H^2_w is singly generated by a polynomial of degree less than N.*

Theorem 2.9. *[SZ02] If w is of type I, then the only minimal reducing subspaces of S in H^2_w are X_n, $0 \leq n \leq N-1$. Moreover, S has exactly 2^N distinct reducing subspaces in H^2_w.*

Theorem 2.10. *[SZ02] If w is of type II, then S has infinitely many distinct minimal reducing subspaces in H^2_w.*

Another significant work in this respect is by Hazarika and Arora in their paper [HA04]. They extended the study of Stessin and Zhu and considered unilateral unweighted operator shifts on an operator weighted sequence space $\ell^2_A(K)$. Here, A is considered as a uniformly bounded sequence of positive invertible self adjoint diagonal operators on a Hilbert space. In their paper, they gave a detailed description of the minimal reducing subspaces of this operator. Below we give a brief discussion of their work.

Let K be a separable Hilbert space with an orthonormal basis $\{e_n\}_{n=0}^{\infty}$. Let $A = \{A_n\}_{n=0}^{\infty}$ be a sequence of bounded linear operators on K. For each $n = 0, 1, 2, \dots,$ A_n is considered as positive, invertible and self adjoint. Also, the sequence of operators $\{A_n^{-1}\}_{n=0}^{\infty}$ is uniformly bounded. Then the operator weighted sequence space is defined as

$$\ell^2_A(K) = \{f = (f_0, f_1, f_2, \dots) : f_i \in K \text{ and } \sum_{k=0}^{\infty} \|A_k f_k\|^2 < \infty\},$$

and the inner product is given as

$$\langle f, g\rangle_A = \sum_{k=0}^{\infty} \langle A_k f_k, A_k g_k\rangle$$

Let S be the (unweighted) unilateral shift operator on $\ell^2_A(K)$. Let $x^i y^j$ be the sequence in K, where e_i is the $(j+1)$th entry and all other entries are zero. A proper look into this sequence shows that $\{x^i y^j\}_{i,j=0}^{\infty}$ is an orthonormal basis for the Hilbert space $\ell^2_A(K)$. The shift operator S can then be written as

$$S(x^i y^j) = x^i y^{j+1}$$

for all $i, j = 0, 1, 2, \dots$. An important thing to be noted in this paper is that the weights A are taken to be positive diagonal operators on the Hilbert space K. And, for each $n = 0, 1, 2, \dots,$ the diagonal elements of the operator A_n are $\{\alpha_i^{(n)}\}_{i=0}^{\infty}$. On the basis of these diagonal elements, which are clearly positive, the weights A are divided into three types:

Type I: A is of type I if for each pair of distinct non negative integers n, m, there is an integer $k > 0$ such that $\frac{\beta_m^{(k)}}{\beta_m^{(0)}} \neq \frac{\beta_n^{(k)}}{\beta_n^{(0)}}$.

Type II: A is of type II if it is not of type I, which means there exists distinct non negative integers n, m, such that $\frac{\beta_m^{(k)}}{\beta_m^{(0)}} \neq \frac{\beta_n^{(k)}}{\beta_n^{(0)}}$ for every positive integer k.

Let X_n be a subspace of $\ell_A^2(K)$ given by $X_n = Span\{x^n y^k : k = 0, 1, 2, \dots\}$. The following lemma shows that X_n is a reducing subspace of S.

Lemma 2.11. *[HA04] For non negative integers i, k*

$$S^*(x^i y^k) := \begin{cases} 0, & \text{if } k = 0; \\ \left(\frac{\alpha_i^{(k)}}{\alpha_i^{(k-1)}}\right)^2 x^i y^{k-1}, & \text{if } k > 0. \end{cases}$$

The concluding results of this paper by Hazarika and Arora are stated below:

Theorem 2.12. *[HA04] X_n's are the only reducing subspaces of the shift S if we take the weight sequence A to be of type I.*

Theorem 2.13. *[HA04] X_n's are not the only reducing subspaces of the shift S when the weight sequence A are of type II. In fact, in this case S may have infinite number of reducing subspaces.*

From the above discussions on the unilateral operator weighted shift, we have seen that while finding out reducing and minimal reducing subspaces of these shifts, various restrictions are imposed on the operator weights that are considered. One such set of conditions is to assume that the weights are self adjoint and invertible. While in another situation, it is assumed that the operator weights are simultaneously diagonalizable i.e, they are mutually commuting. The following work done by Hazarika and Gogoi in their paper [HG17a] gave some significant results on the reducing and minimal reducing subspaces of unilateral operator weighted shift, where the operator weights are not necessarily simultaneously diagonalizable. Below we mention some important results in [HG17a].

We begin with a brief introduction to the weights considered in [HG17a]:

Let $\mathcal{B}(K)$ denote the set of all bounded linear operators on the separable complex Hilbert space K with orthonormal basis $\{e_n\}_{n=0}^{\infty}$, and $\mathcal{T}$ be the subset of $\mathcal{B}(K)$ defined as follows:

$\mathcal{T} := \{T \in \mathcal{B}(K) \big|$ T is invertible in $\mathcal{B}(K)$ and the matrix of T with respect to $\{e_n\}_0^{\infty}$ has exactly one non zero entry in each row and each column.$\}$

We observe the following:
(i) If $T_1, T_2 \in \mathcal{T}$, then $T_1 T_2 \in \mathcal{T}$. However, T_1 and T_2 need not commute and hence elements of $\mathcal{T}$ are not simultaneously diagonalizable with respect to $\{e_n\}_0^{\infty}$.
(ii) If $T \in \mathcal{T}$ then its Hilbert adjoint T^* and inverse T^{-1} are also in $\mathcal{T}$.

(iii) Elements of $\mathcal{T}$ need not be self adjoint or normal.

The unilateral operator weighted shift W on $\ell^2_+(K)$ is then considered with uniformly bounded weights $\{A_n\}_{n\in\mathbb{N}_0}$ in $\mathcal{T}$.

Definition 2.14. *[HG17a] Let $\mathcal{S}$ be the vector space of all finite linear combinations of finite products of W and W^*. For non-zero $F \in \ell^2_+(K)$, let $\mathcal{S}F := \{TF : T \in \mathcal{S}\}$. Then the closure of $\mathcal{S}F$ in $\ell^2_+(K)$ is a reducing subspace of W, denoted by X_F. Clearly X_F is the smallest reducing subspace of W in $\ell^2_+(K)$ containing F.*

One of the first results that is established in [HG17a] is the following:

Theorem 2.15. *[HG17a] Let $\{A_n\}_{n=0}^{\infty}$ be a sequence in $\mathcal{T}$ and $\sup_n \|A_n\| < \infty$. Then there exists a sequence $B = \{B_n\}_{n=0}^{\infty}$ of positive invertible diagonal bounded linear operators on K such that the operator weighted shift W on $\ell^2_+(K)$ with weight sequence $\{A_n\}_{n=0}^{\infty}$ is unitarily equivalent to the unilateral shift S on $\ell^2_B(K)$.*

We have already seen that in [HA04], the minimal reducing subspaces of S on $\ell^2_B(K)$ is determined when B represents a uniformly bounded sequence of invertible diagonal operators on K. So in view of the work done in [HA04], we should also be able to determine the minimal reducing subspaces of the operator weighted shift W on $\ell^2_+(K)$ with weights $\{A_n\}$ in $\mathcal{T}$. However, this is not an easy task because of the complex transformations involved in the process. It becomes quite difficult to easily appreciate the end result. Hence, a different approach in adopted in this work.

Here, the unilateral operator weighted shift W with non diagonal operator weights is represented as a direct sum of scalar weighted shift operators, as suggested in [Pil80]. In this respect, the Theorem 3.9 [Lam71] is stated below for reference.

Theorem 2.16. *[Lam71] The operator weighted shift W on $\ell^2(K)$ with operator weights $\{A_n\}_{n=0}^{\infty}$ is a direct sum of scalar weighted shifts if and only if the weakly closed $*$ algebra generated by $\{I, A_0, A_1, \dots\}$ is diagonalizable.*

In view of the above theorem, the operator weighted shift W on $\ell^2_+(K)$ with weights A_n in $\mathcal{T}$ can be expressed as a direct sum of scalar weighted shift operators.

Theorem 2.17. *[HG17a] Let W be an operator weighted shift on $\ell^2_+(K)$ with uniformly bounded operator weights $\{A_n\}_{n\in\mathbb{N}_0}$ where each $A_n \in \mathcal{T}$. Then there exists scalar weighted shift operators S_0, S_1,... on ℓ^2 such that W on $\ell^2_+(K)$ is unitarily equivalent to $S_0 \oplus S_1 \oplus \dots$ on $\ell^2 \oplus \ell^2 \oplus \dots$.*

Based on these scalar weighted shifts, the unilateral operator weighted shift is classified into three types:

Type I: If no two scalar shifts S_n's are identical.

Type II: At least two distinct scalar shifts S_n and S_m are identical.

Type III: For $n, m = 0, 1, 2, \dots$, n is said to be related to m with respect to W, denoted by $n \sim^W m$ if S_n and S_m are identical. Clearly $\sim^W$ is an equivalence relation on $\mathbb{N}_0$. W is said to be of Type III if $\sim^W$ partitions $\mathbb{N}_0$ into a finite number of equivalence classes.

Some of the significant results on reducing and minimal reducing subspaces of the unilateral operator weighted shift W are mentioned below:

Theorem 2.18. *[HG17a] If W is of Type I, then $X_{g_{n,0}}$ for $n \in \mathbb{N}_0$ are the only minimal reducing subspaces of W in $\ell^2_+(K)$.*

Theorem 2.19. *[HG17a] If W is of Type II, then W has minimal reducing subspaces other than $X_{g_{n,0}}$ ($n \in \mathbb{N}_0$). In fact, for every W-transparent F, X_F is a minimal reducing subspace and hence W will have infinitely many minimal reducing subspaces in $\ell^2_+(K)$. (A linear expression $F = \sum_{i\in\mathbb{N}_0} \alpha_i g_{i,0}$ in $\ell^2_+(K)$ is said to be W-transparent if for every pair of non-zero scalars α_i and α_j, we have $i \sim^W j$.)*

Theorem 2.20. *[HG17a] If W is of Type III, then every reducing subspace of W must contain a minimal reducing subspace.*

Finally, we discuss the paper [HG17b] by Hazarika and Gogoi, where the unilateral (unweighted) shift S is considered on the operator weighted sequence space $\ell^2_B(K)$. Here, the set of operator weights $B = \{B_n\}_{n\in\mathbb{N}_0}$ are in $\mathcal{T}$ (as given in [HG17a]).

Let $B = \{B_n\}_{n\in\mathbb{N}_0}$ be a uniformly bounded sequence of operators in the class $\mathcal{T}$. For each $n \in \mathbb{N}_0$ let $\gamma_j^{(n)}$ denote the unique non zero entry occurring in the j^{th} column of the matrix of B_n. The weights $\{B_n\}$ are then divided into three types:

Type I: If for each pair of distinct non negative integers m and n there exist some positive integer k such that $\frac{\gamma_m^{(k)}}{\gamma_m^{(0)}} \neq \frac{\gamma_n^{(k)}}{\gamma_n^{(0)}}$.

Type II: If there exist distinct non negative integers m and n such that $\frac{\gamma_m^{(k)}}{\gamma_m^{(0)}} = \frac{\gamma_n^{(k)}}{\gamma_n^{(0)}}$ for every positive integer k.

For $i, j \in \mathbb{N}_0$, let $f_{i,j}$ in $\ell^2_B(K)$ be the sequence that has e_i as the j^{th} entry and zero as all other entries. It is proved in [HG17b] that $\{f_{i,j}\}_{i,j\in\mathbb{N}_0}$ is an orthonormal basis for the Hilbert space $\ell^2_B(K)$. Hence the unilateral shift S can be written as $Sf_{i,j} = f_{i,j+1}$.

Let $\mathcal{S}$ be the collection of all finite linear combinations of finite products of the operators S and its adjoint. Define $\mathcal{S}F = \{TF : T \in \mathcal{S}\}$, where F is a non zero function in $\ell^2_B(K)$. The closure of the space $\mathcal{S}F$ in $\ell^2_B(K)$, denoted by X_F is a reducing subspace of S. It can be clearly seen that X_F is the smallest reducing subspace of $\ell^2_B(K)$ that contains F.

Based on this classification, the main results in this paper are the following:

Theorem 2.21. *[HG17b] Let $B = \{B_n\}_{n\in\mathbb{N}_0}$ be a uniformly bounded sequence of operators in $\mathcal{T}$, and S be the unilateral shift on $\ell^2_B(K)$. If the weight sequence $\{B_n\}_{n\in\mathbb{N}_0}$ is of type I, then $X_{f_{n,0}}$ for $n \in \mathbb{N}_0$ are the only minimal reducing subspaces of S in $l^2_B(K)$.*

Theorem 2.22. *[HG17b] Let $B = \{B_n\}_{n\in\mathbb{N}_0}$ be a uniformly bounded sequence of operators in $\mathcal{T}$, and S be the unilateral shift on $l^2_B(K)$. If $\{B_n\}_{n\in\mathbb{N}_0}$ is of type II, then S has minimal reducing subspaces other than $X_{f_{n,0}}$, $n \in \mathbb{N}_0$.*

3 Conclusion

From the above discussion, we have seen that operator weighted shifts have been very widely studied and most of these work vary on the basis of the different types of operator weights that have been considered. So, keeping in view the fact that we still have quite a number of restrictions being imposed on these operator weights, there is a lot of scope for research in this area. As for example, the operator weights taken in [HG17a] and [HG17b] are neither normal nor simultaneously diagonalizable, but still there is a certain specific structure to its matrix representation. We cannot say anything if such a structure is not maintained. Also, in recent times we have seen that certain graph theoretical concepts are introduced and weighted shifts are defined on directed trees. In this context, we can refer the following papers: [JS12], [MS16], [SW89]. Although many important properties such as subnormality and hyponormality have already been discussed in this regard, but still very less can be seen about reducing subspaces of such shifts defined in directed trees.

4 Bibliography

[Beu49] A. Beurling. On two problems concerning linear transformations in hilbert spaces. *Acta Math.*, 81, 1949.

[Bou06] A. Bourhim. Spectrum of bilateral shifts with operator-valued weights. *Proc. Amer. Math. Soc.*, 134(7):2131–2137, 2006.

[Fil] P. A. Fillmore. Notes on operator theory. *Van Nostrand Reinhold Co.*

[Gel69] R. Gellar. Operators commuting with a weighted shift. *Proc. Amer. Math. Soc.*, 23(3):538–545, 1969.

[Guy85] J. Guyker. On reducing subspaces of normally weighted bilateral shifts. *Houston J. Math.*, 11(4):515–521, 1985.

[HA04] M. Hazarika and S. C. Arora. Minimal reducing subspaces of the unilateral shift operator on an operator weighted sequence space. *Indian J. Pure Appl. Math.*, 35(6):747–757, 2004.

[Her90] D. Herrero. Spectral pictures of hyponormal bilateral operator weighted shifts. *Proc. Amer. Math. Soc.*, 109(3):753–763, 1990.

[HG17a] M. Hazarika and P. S. Gogoi. Minimal reducing subspaces of an operator-weighted shift. *Ann. Funct. Anal.*, 8(4):531–546, 2017.

[HG17b] M. Hazarika and P. S. Gogoi. Minimal reducing subspaces of the unilateral shift. *East-West J. of Mathematics*, 19(1):43–51, 2017.

[Jab04] Z. J. Jabłoński. Hyperexpansive operator valued unilateral weighted shifts. *Glasg. Math. J.*, 46(2):405–416, 2004.

[JS12] Jung I. B. Jabłoński, Z. J. and J. Stochel. Weighted shifts on directed trees. *Mem. Amer. Math. Soc.*, 216(1017), 2012.

[JW07] Jin Y. F. Jiang, C. L. and Z. Y. Wang. About operator weighted shift. *Acta. Math. Sin.-English Ser.*, 23(8):1385–1390, 2007.

[Lam71] A. Lambert. Unitary equivalence and reducibility of invertibly weighted shifts. *Bull. Austral. Math. Soc.*, 5, 1971.

[LS01] Ji Y. Q. Li, J. X. and S. L. Sun. The essential spectrum and banach reducibility of operator weighted shifts. *Acta. Math. Sin.-English Ser.*, 17(3):413–424, 2001.

[MS16] W. Majdak and J.B. Stochel. Weighted shifts on directed semi-trees: an application to creation operators on segal-bargmann spaces. *Complex Anal. Oper. Theory*, 10(7):1427–1452, 2016.

[Nik] N. K. Nikol'skii. Treatise on the shift operator: spectral function theory. *Springer-Verlag.*

[Nik67] N. K. Nikol'skii. Invariant subspaces of weighted shift operators. *Math. USSR. Sbornik*, 3(2):159–176, 1967.

[Pil80] V. S. Pilidi. On invariant subspaces of multiple weighted shift operators. *Math. USSR Izvestija*, 14(2):345–365, 1980.

[PP94] C. M. Pearcy and S. Petrovic. On polynomially bounded weighted shifts. *Houston J. Math.*, 20(1):27–45, 1994.

[Sar74] D. Sarason. Invariant subspaces. *Topics in Operator Theory, Math surveys*, 13, 1974.

[Shi74] A. L. Shields. Weighted shift operators and analytic function theory. *Topics in operator theory, Math surveys*, 13, 1974.

[SW89] Fujii M. Sasaoka, H. and Y. Watatani. Adjacency operators of infinite directed graphs. *Mathematica Japonica*, 34, 1989.

[SZ02] M. Stessin and K. Zhu. Reducing subspaces of weighted shift operators. *Proc. Amer. Math. Soc.*, 130(9):2631–2639, 2002.

Risk Measures

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Abstract. In financial market, a risk measure is used to determine the amount of capital to be kept in reserve. The purpose of this reserve is to make the risks taken by financial institutions, such as banks and insurance companies, acceptable to the regulator. In this article we try to review various risk measures and its properties available in the literature which are used to estimate the market risk.

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1 Introduction

A major concern for the regulators and owners of financial institutions is the market risk of a portfolio consisting of risky assets, e.g. a stock market index or a mutual fund, and the adequacy of capital to meet such risk (see [DDV00]). Market risk is the risk of losses in positions arising from the movements in market prices of assets in a portfolio. "The financial disasters of the early 1990s incurred by several institutions such as Orange County, Procter and Gamble and NatWest, through inappropriate derivatives pricing and management, as well as fraudulent cases such as Barings Bank and Sumitomo, have brought risk management and regulation of financial institutions to the forefront of policy making and public discussions" (see [DDV00]). These disasters proved that billions of dollars can be lost because of poor supervision and management of financial risks. "The notion of risk measure arose from the method of quantifying risk" (see [Tsu09]). In financial market, a risk measure is used to determine the amount of capital to be kept in reserve.

There are several ways of measuring the potential loss amount due to market risk. Value at risk (VaR), was developed to measure financial market risk in response to the financial disasters of the early 1990s (see [Jor00]). Since then VaR has spread well beyond derivatives and is changing the way institutions approach their financial risk (see [Jor00]). VaR is an extreme quantile of the marginal loss distribution. Its use was recommended by the Basel Committee on Banking Supervision in 1996. But VaR is not a coherent risk measure. In recent years attention has turned towards convex and coherent risk measures. Artzner et al. ([Art97, ADEH99]) introduced the concept of coherent risk measure which is a function that satisfies monotonicity, subadditivity, homogeinity and translational invariance. VaR does not satisfy the subadditivity property. More importantly, VaR is not able to distinguish portfolios which bear different levels of risk (see [ANS01]). To address these issues, expected shortfall was introduced by Artzner et al., which is a coherent risk measure (see [ADEH99]). ES is defined as the mean of the conditional log return distribution, given the event that the log return is less than the VaR. It is closely linked to VaR, and is regarded as a good supplement to the VaR (See [ANS01]). Another coherent risk measure called Median Shortfall(MS) was introduced by So and Wong(see [SW12]). MS is the median loss when the loss in the investment exceeds the VaR level. There are several other risk measures to estimate market risk which we shall discuss in the next sections.

In section 2 we give the definition of risk measure along with its properties. In section 3 we discuss about the various types of risk measures which are used to estimate market risk. And in section 4 we give the summary.

2 Risk measure

A risk measure is a function that assigns real numbers to the possible outcomes of a random financial quantity, such as an insurance claim or loss of a portfolio (see [BJPZ08]). Loss due to price fluctuations or insurance claim size are usually represented by random variables. Let ψ denote the set of real valued random variables on a probability space (Ω, $\mathcal{F}$, $\mathbb{P}$).

Definition 2.1. *(Delbaen [Del02]) A risk measure ρ is a mapping from ψ to $\mathbb{R}$ satisfying certain properties, viz.*

1. $X \geq 0 \Rightarrow \rho(X) \leq 0$.
2. $X \geq Y \Rightarrow \rho(X) \leq \rho(Y),\ X, Y \in \psi$.
3. $\rho(\lambda X) = \lambda\rho(X),\ \forall\lambda \geq 0,\ X \in \psi$.
4. $\rho(X + k) = \rho(X) - k,\ \forall k \in \mathbb{R},\ X \in \psi$.

The term "coherent" risk measure is reserved for risk measures that satisfies one more additional property, viz. subadditivity. Artzner et al. introduced the concept of coherent risk measure (see [Art97, ADEH99]).

Definition 2.2. *(Delbaen [Del02]) A risk measure ρ on ψ is said to be coherent if in addition to the properties $1 - 4$, ρ also satisfies the following "subadditivity" property, viz.*

$$\rho(X + Y) \leq \rho(X) + \rho(Y), \forall X,\ Y \in \psi.$$

For a nice representation theorem, the following continuity property is needed:

- **Fatou property**: If the X_n is uniformly bounded in absolute value by 1 and $X_n \xrightarrow{\mathbb{P}} X$, then $\rho(X) \leq \liminf \rho(X_n)$.

Delbaen [Del02] proved that any coherent risk measure with Fatou property can be represented as

$$\rho(X) = \sup\{E_Q(X) : Q \in \mathbb{L}\},$$

where $\mathbb{L}$ is a set of probability measures and each member of $\mathbb{L}$ is absolutely continuous with respect to P.

Jouini, Schachermayer, and Touzi [JST06] proved that for coherent risk measures, the Fatou property automatically follows from the law invariance under the assumption that $(\Omega,\ \mathcal{F},\ \mathbb{P})$ is atomless. Kusuoka [Kus01] showed that every law invariant, comonotonically additive coherent risk measure can be represented as the expectation of the risk under a convexly distorted distribution. Föllmer and Schied [FS02] and Fritelli and Rosazza Gianin [FG02] generalized coherent risk measures to the convex case by replacing the two properties of subadditivity and positive homogeneity with the property of convexity. They showed that, as with coherent risk measures, each convex lower semi-continuous risk measure admits a dual representation. In the next section we shall discuss several coherent and convex risk measures.

3 Well known risk measures

- **Value-at-risk (VaR):** VaR is a popular measure of market risk associated with an asset or portfolio of assets (see [Hul12]). It is defined as an extreme quantile of the marginal loss distribution. It is a cut-off value that separates future loss events into risky and non-risky scenarios (see [SW12]). VaR's use was recommended by the Basel Committee on Banking Supervision in 1996 as a benchmark risk measure and has been widely used by financial institutions for asset management and minimization of risk. Comprehensive discussions on VaR are available in Duffie and Pan [DP97], Danielsson and De Vries [DDV00] and Jorion [Jor00].

  Let, X be a random variable representing a loss of some financial position. For $0 < p < 1$, the $(1-p)$th quantile of the distribution with distribution function F is defined as

  $$Q_p = \inf\{x : F(x) \geq (1-p)\},$$

  the $100(1-p)$ percent VaR, denoted by VaR_p, is the negative $(1-p)$th quantile of the marginal distribution of X, i.e.

  $$VaR_p = -Q_p. \tag{38}$$

  Hence estimation of VaR_p essentially reduce to the problem of estimation of the quantile Q_p.

  VaR_p satisfies the properties $1-4$ in Definition (2.1) (see [ADEH99]). But it fails to satisfy the "subadditivity" property. Hence, VaR is not a coherent risk measure. This implies that the risk of a portfolio, when measured by VaR, can be larger than the sum of the standalone risks of its components.

- **Expected shortfall (ES):** Artzner et al. ([Art97, ADEH99]) have shown that VaR does not provide any information about the size of the potential loss when it exceeds the VaR level. *ES is defined as the mean of the conditional return distribution, given the event that the return is less than the VaR.* For a given level p, the shortfall distribution is given by the cumulative distribution function Θ_p defined by:

  $$\Theta_p(x) = P\{X \leq x | X > Q_p\}.$$

The mean of this distribution is called the expected shortfall, and is denoted by ES_p. Mathematically, the $100(1-p)$ ES can be written as

$$ES_p = -\frac{1}{p}\int_{1-p}^{1} Q_p(u)du.$$

It is closely linked to VaR, and is regarded as a good supplement to the VaR (See [Ace02, Ace03]). ES is a coherent risk measure (see [ADEH99]). Yamai and Yoshiba showed that ES is easily decomposed and optimized, while VaR is not (see [YY02]). The decomposition of risk is a useful tool for managing portfolio risk (see [YY02]). For example, risk decomposition enables risk managers to select assets that provide the best risk-return trade-off, or to allocate Ã¯Â¿Â½economic capitalÃ¯Â¿Â½ to individual risk factors (see [YY02]). The concept of VaR decomposition was proposed by Garman in 1997 (see [Gar97]). Yamai and Yoshiba described the method of decomposing VaR and ES which was developed by Hallerbach in 1999 and Tasche in 2000 (see [Hal99, Tas99]). One disadvantage of ES is that it is not elicitable whereas VaR is elicitable [Gne11].

- **Median shortfall (MS):** So and Wong introduced this risk measure and named it Median Shortfall(MS) (see [SW12]). By definition, MS is the median of the conditional return distribution, given that the return is less than the VaR level (see [SW12]). Let Θ_p denote the distribution function of this conditional return distribution. It is defined as follows

$$\Theta_p(x) = P\{X \le x | X > Q_p\}.$$

The median of this distribution is called the Median Shortfall, denoted by MS_p (see [SW12]). The MS can be written as

$$MS_p = -\inf\{x : \Theta_p(x) \ge 0.5\} = -Q_{0.5p}. \tag{39}$$

The marginal loss distribution F and the quantile function Q are unknown. Therefore VaR_p, ES_p and MS_p are unknown in practice. From (38) and (39), we see that estimation of VaR_p and MS_p essentially reduce to the problem of estimation of the quantiles of the marginal loss distribution.

- **Entropic risk measure:** The entropic risk measure [FK11] is defined as

$$\begin{aligned} e_\gamma &:= \frac{1}{\gamma} log E_{\mathbb{P}}[e^{-\gamma X}] \\ &= \sup_Q E_Q[-X] - \frac{1}{\gamma} H(Q|\mathbb{P}). \end{aligned}$$

for parameters $\gamma \in [0, \infty)$, where $e_0 := E_{\mathbb{P}}[-X]$ and $H(Q|\mathbb{P})$ denotes the relative entropy of Q with respect to P. "VaR is the one which is used most widely, but it has various deficiencies; in particular it is not convex and may thus penalize a desirable diversification" [FK11]. ES is a coherent risk measure, i. e., convex and also positively homogeneous. As shown by Kusuoka [Kus01] in the coherent and by Frittelli and Rosazza Gianin [FG02] in the general convex case, ES is a basic building block for any law-invariant convex risk measure. The entropic risk measures e_γ are convex, and they are additive for independent positions. The coherent version of the entropic risk measure was discussed by Föllmer and Knispel in 2011 (see [FK11]).

Definition 3.1. *For each $c > 0$, the risk measure ρ_c defined by*

$$\rho_c = \sup_{Q \in \mathbb{M}_1 : H(Q|P) \leq c} E_Q(-X), \; X \in L^\infty$$

is called the coherent entropic risk measure.

Föllmer and Knispel [FK11] also showed that ρ_c is law-invariant. They have also showed the relation between coherent entropic risk measure and convex entropic risk measure.

- **Entropic Value-at-Risk (EVaR):** Ahmadi-Javid in 2012 proposed a new coherent risk measure which is the possible upper bound obtained from the Chernoff inequality for the VaR (see [AJ12b]). Let, L_M be the set of all Borel Measurable functions whose moment generating function is $M_X(z) = E(e^{zX})$, for all $z \in \mathbb{R}$ and L_{M^+} be the set of all Borel Measurable functions whose moment generating function $M_X(z)$ exists for all $z \geq 0$. The Chernoff inequality [C52] for any constant a and $X \in L_{M^+}$ is as follows:

$$P(X \geq a) \leq e^{-za} M_X(z), \; \forall z > 0.$$

By solving the equation $e^{-za} M_X(z) = p$ with respect a for $p \in]0, 1]$, we obtain

$$a_X(p, z) := z^{-1} ln(M_X(z)/p),$$

for which we have $P(X \geq a_X(p, z)) \leq p$. In fact, for each $z > 0$, $a_X(p, z)$ is an upper bound for VaR_p. The author considered the best upper bound of this type as a new risk measure that bounds VaR_p by using exponential moments. The special case $a_X(1, z)$ is known as the *exponential premium* in the actuarial literature (see [GDVH84, Ger74]). In finance literature [FS02], it is considered as a convex risk measure, which is called the entropic risk measure.

Definition 3.2. *(Ahmadi-Javid [AJ12b]) The entropic value-at-risk (EVaR) of $X \in L_{M^+}$ with confidence level $1 - p$ is defined as:*

$$EVaR_p := \inf_{z>0} \{a_X(p, z)\} = \inf_{z>0} \{z^{-1} ln(M_X(z)/p)\}. \tag{40}$$

Ahmadi-Javid [AJ12b] proved that the EVaR is a coherent risk measure. Ahmadi-Javid [AJ12b] also showed that EVaR is more risk-averse as compared to the ES at the same confidence level. Hence, the EVaR proposes a financial and insurance agency distributing more assets to avoid risk. Inspired by the dual representation of the EVaR, which is closely related to the Kullback-Leibler [KL51] divergence, also known as the relative entropy, Ahmadi-Javid [AJ12b] defined a large class of coherent risk measures, called g-entropic risk measures.

The generalized relative entropy of Q with respect to $\mathbb{P}$, denoted by $H_g(\mathbb{P}, Q)$, is an information-type pseudo-distance or divergence measure from Q to $\mathbb{P}$:

$$H_g(\mathbb{P}, Q) := \int g\left(\frac{dQ}{d\mathbb{P}}\right) d\mathbb{P},$$

where g is a convex function with $g(1) = 0$ (see [AJ12b]). This quantity is an important non-symmetric divergence measure discussed in (see [AS66]). Note that $H_g(\mathbb{P}, Q) \geq 0$, and $H_g(\mathbb{P}, Q) = 0$ if $Q = \mathbb{P}$.

Definition 3.3. *(Ahmadi-Javid [AJ12b]) Let g be a convex function with $g(1) = 0$, and β be a nonnegative number. The g-entropic risk measure with divergence level β is defined as*

$$ER_{g,\beta}(X) := \sup_{Q \in \Im} E_Q(X),$$

where $\Im = \{Q << \mathbb{P} : H_g(\mathbb{P}, Q) \leq \beta\}$.

Ahmadi-Javid [AJ12a] shows how this risk measure can be used in machine learning when uncertainty affects the input data. Ahmadi-Javid and Fallah-Tafti [AJFT19] has done portfolio optimization with EVaR.

- **Spectral risk measures (SRMs):** Spectral risk measures proposed by Acerbi ([Ace02, Ace03]) belong to the family of coherent risk measures and hence inherit the properties of such measures. One of the nice features of SRMs is that they relate the risk measure to the user's risk aversion in effect, the spectral risk measure is a weighted average of the quantiles of a loss distribution, the weights of which depend on the user's risk aversion (see [DCS08]). SRMs therefore enable us to link the risk measure to the userâ€™s attitude towards risk, and we might expect that if a user is more risk-averse, other things being equal, then that user should face a higher risk, as given by the value of the SRM (see [DCS08]). "A user who is risk-averse might prefer to work with a risk measure that takes account of his/her risk aversion, and this takes us to the class of spectral risk measures (SRMs)" (see [DCS08]). The SRMs are defined by a general convex combination of ES.

Definition 3.4. *(Gzyl and Mayoral [HS08]) An element $\phi \in \mathfrak{L}_1([0,1])$ is called an admissible risk spectrum if*

1. $\phi \geq 0$
2. $\int_0^1 |\phi(t)|dt = 1$
3. *ϕ is non-increasing.*

Definition 3.5. *(Gzyl and Mayoral [HS08]) Let, an admissible risk measure $\phi \in \mathfrak{L}_1([0,1])$, then the spectral risk measure is defined by*

$$M_\phi = -\int_0^1 \phi(u) Q_u du. \tag{41}$$

ϕ is called the Risk Aversion Function. The Risk Aversion Function is defined by Cotter and Dowd [CD06]

$$\phi(u) = \frac{ke^{-k(1-u)}}{1 - e^{-k}} \tag{42}$$

where $k \in (0, \infty)$ is the user's coefficient of absolute risk aversion. Dowd et al. [DCS08] proposed two more Risk Aversion Functions. The authors showed that SRMs can also be obtained based on other Risk Aversion Functions. These are called power spectral risk measures (PSRMs). These are

$$\phi(u) = \begin{cases} \gamma(1-u)^{\gamma-1} & \text{for} \quad \gamma < 1 \\ \gamma u^{\gamma-1} & \text{for} \quad \gamma > 1. \end{cases}$$

From equation (42) we see that if $\phi(u) = \frac{1}{p}1_{0\geq u\geq p}$ then M_ϕ is defined as the ES which is a spectral risk measure (see [HS08]). But VaR is not a spectral risk measure as it is not a coherent risk measure. Gzyl and Mayoral [HS08] studied the relationship between SRMs and distortion risk measures. The authors also proved that SRMs are equivalent to distorted risk pricing measures, or equivalently, spectral risk functions are related to distorted functions. It is observed that SRMs are not elicitable but Ziegel has proved that under one condition SRM is elicitable [Zie16].

- **Distortion risk measure:** Using Kusuoka [Tsu09] representation the class of distortion risk measure can be written in the following form

$$\rho_D = \int_{[0,1]} Q_u dD(u) = \int_{\mathcal{R}} x dD \circ F(x), \tag{43}$$

where D is a convex distortion function and Q_u is the quantile function. The SRMs and weighted VaR [Che06] are same class of risk measures.

Definition 3.6. *[Tsu13] A distortion function D is a right-continuous increasing function on $[0,1]$ with values in $[0,1]$ such that $D(0) = 0$ and $D(1) = 1$.*

If F is a distribution function and D a distortion then $R = D \circ F$ is again a distribution function, and it is called the distorted distribution under D. A distortion can be viewed as a transformation on the space of distribution function's. And the expectation under the distorted distribution function R, that is,$\int xdR(x)$, will be called the distorted expectation under D. A distorted expectation is also called the Choquet integral [Cho54], the nonadditive integral [Den13], or the fuzzy integral [GSM10]. Tsukahara [Tsu09] did not use these terms because the distorted expectation is a proper integral and distribution function is being distorted, not a probability measure.

Tsukahara [Tsu13] suggested a one-parameter family of distortion that yields several classes of coherent risk measures

- Proportional hazards (PH) distortion (Wang, [Wan96]):$D_\theta^{PH}(u) = 1 - (1-u)^\theta$,
- Proportional odds (PO) distortion: $D_\theta^{PO}(u) = \theta u/[1-(1-\theta)u]$,
- Gaussian distortion (Wang, [Wan00]): $D_\theta^{GA}(u) = \Phi(\Phi^{-1}(u) + log\theta)$

4 Summary

The above mentioned risk measures are the well known risk measures to estimate the market risk. There are a lot of literature available to estimate these risk measures. Especially the VaR, ES and MS. Various authors have studied the estimation procedures of these risk measures. But we find a very few literature on the estimation methods of spectral risk measure, distortion risk measure and convex risk measures. Other than empirical distribution function we do not find any other distribution function estimator considered in the estimation of SRMs and distortion risk measures. As these risk measures are an important tool to estimate the market risk we should concentrate more on the estimation methods of these risk measures. As there are already a lot of literature regarding the properties of these risk measures. So further studies should be concentrated more on estimation methods of these risk measures.

5 Bibliography

[Ace02] Carlo Acerbi. Spectral measures of risk: A coherent representation of subjective risk aversion. *Journal of Banking & Finance*, 26(7):1505–1518, 2002.

[Ace03] Carlo Acerbi. Coherent representations of subjective risk aversion. *Risk Measures for the 21st Century*, page 147, 2003.

[ADEH99] Philippe Artzner, Freddy Delbaen, Jean-Marc Eber, and David Heath. Coherent measures of risk. *Mathematical finance*, 9(3):203–228, 1999.

[AJ12a] Amir Ahmadi-Javid. Application of entropic value-at-risk in machine learning with corrupted input data. In *2012 11th International Conference on Information Science, Signal Processing and their Applications (ISSPA)*, pages 1104–1107. IEEE, 2012.

[AJ12b] Amir Ahmadi-Javid. Entropic value-at-risk: A new coherent risk measure. *Journal of Optimization Theory and Applications*, 155(3):1105–1123, 2012.

[AJFT19] Amir Ahmadi-Javid and Malihe Fallah-Tafti. Portfolio optimization with entropic value-at-risk. *European Journal of Operational Research*, 2019.

[ANS01] Carlo Acerbi, Claudio Nordio, and Carlo Sirtori. Expected shortfall as a tool for financial risk management. *arXiv preprint cond-mat/0102304*, 2001.

[Art97] Philippe Artzner. Thinking coherently. *Risk*, pages 68–71, 1997.

[AS66] Syed Mumtaz Ali and Samuel D Silvey. A general class of coefficients of divergence of one distribution from another. *Journal of the Royal Statistical Society: Series B (Methodological)*, 28(1):131–142, 1966.

[BJPZ08] Vytaras Brazauskas, Bruce L Jones, Madan L Puri, and Ričardas Zitikis. Estimating conditional tail expectation with actuarial applications in view. *Journal of Statistical Planning and Inference*, 138(11):3590–3604, 2008.

[C52] Herman Chernoff et al. A measure of asymptotic efficiency for tests of a hypothesis based on the sum of observations. *The Annals of Mathematical Statistics*, 23(4):493–507, 1952.

[CD06] John Cotter and Kevin Dowd. Extreme spectral risk measures: an application to futures clearinghouse margin requirements. *Journal of Banking & Finance*, 30(12):3469–3485, 2006.

[Che06] Alexander S Cherny. Weighted v@ r and its properties. *Finance and Stochastics*, 10(3):367–393, 2006.

[Cho54] Gustave Choquet. Theory of capacities. In *Annales de l'institut Fourier*, volume 5, pages 131–295, 1954.

[DCS08] Kevin Dowd, John Cotter, and Ghulam Sorwar. Spectral risk measures: properties and limitations. *Journal of Financial Services Research*, 34(1):61–75, 2008.

[DDV00] J. Danielsson and C. G. De Vries. Value-at-risk and extreme returns. *Annals of Economics and Statistics*, 60:239–270, 2000.

[Del02] Freddy Delbaen. Coherent risk measures on general probability spaces. In *Advances in finance and stochastics*, pages 1–37. Springer, 2002.

[Den13] Dieter Denneberg. *Non-additive measure and integral*, volume 27. Springer Science & Business Media, 2013.

[DP97] Darrell Duffie and Jun Pan. An overview of value at risk. *Journal of derivatives*, 4(3):7–49, 1997.

[FG02] Marco Frittelli and Emanuela Rosazza Gianin. Putting order in risk measures. *Journal of Banking & Finance*, 26(7):1473–1486, 2002.

[FK11] Hans Föllmer and Thomas Knispel. Entropic risk measures: Coherence vs. convexity, model ambiguity and robust large deviations. *Stochastics and Dynamics*, 11(02n03):333–351, 2011.

[FS02] Hans Föllmer and Alexander Schied. Convex measures of risk and trading constraints. *Finance and stochastics*, 6(4):429–447, 2002.

[Gar97] Mark Garman. Taking var to pieces. *Risk*, 10(10):70–71, 1997.

[GDVH84] Marc J Goovaerts, Florent De Vylder, and Jean Haezendonck. *Insurance premiums: theory and applications.* North-Holland Amsterdam, 1984.

[Ger74] Hans U Gerber. On additive premium calculation principles. *ASTIN Bulletin: The Journal of the IAA*, 7(3):215–222, 1974.

[Gne11] Tilmann Gneiting. Making and evaluating point forecasts. *Journal of the American Statistical Association*, 106(494):746–762, 2011.

[GSM10] Michel Grabisch, Michio Sugeno, and Toshiaki Murofushi. *Fuzzy measures and integrals: theory and applications.* Heidelberg: Physica, 2000., 2010.

[Hal99] Winfried G Hallerbach. Decomposing portfolio value-at-risk: A general analysis. Technical report, Tinbergen Institute Discussion Paper, 1999.

[HS08] Gzyl Henryk and Mayoral Silvia. On a relationship between distorted and spectral risk measures. *Revista de economía financiera*, 15:8–21, 2008.

[Hul12] John Hull. *Risk management and financial institutions,+ Web Site*, volume 733. John Wiley & Sons, 2012.

[Jor00] Philippe Jorion. Value at risk. 2000.

[JST06] Elyès Jouini, Walter Schachermayer, and Nizar Touzi. Law invariant risk measures have the fatou property. In *Advances in mathematical economics*, pages 49–71. Springer, 2006.

[KL51] Solomon Kullback and Richard A Leibler. On information and sufficiency. *The annals of mathematical statistics*, 22(1):79–86, 1951.

[Kus01] Shigeo Kusuoka. On law invariant coherent risk measures. In *Advances in mathematical economics*, pages 83–95. Springer, 2001.

[SW12] M. K. P. So and C. M. Wong. Estimation of multiperiod expected shortfall and median shortfall for risk management. *Quantitative Finance*, 12(5):739–754, 2012.

[Tas99] Dirk Tasche. Risk contributions and performance measurement. *Report of the Lehrstuhl für mathematische Statistik, TU München*, 1999.

[Tsu09] Hideatsu Tsukahara. One-parameter families of distortion risk measures. *Mathematical Finance*, 19(4):691–705, 2009.

[Tsu13] Hideatsu Tsukahara. Estimation of distortion risk measures. *Journal of Financial Econometrics*, 12(1):213–235, 2013.

[Wan96] Shaun Wang. Premium calculation by transforming the layer premium density. *ASTIN Bulletin: The Journal of the IAA*, 26(1):71–92, 1996.

[Wan00] Shaun S Wang. A class of distortion operators for pricing financial and insurance risks. *Journal of risk and insurance*, 67(1):15–36, 2000.

[YY02] Y. Yamai and T Yoshiba. Comparative analysis of expected shortfall and value-at-risk: their estimation error, decomposition, and optimization. *Monetary and Economic Studies*, pages 87–122, 2002.

[Zie16] Johanna F Ziegel. Coherence and elicitability. *Mathematical Finance*, 26(4):901–918, 2016.

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