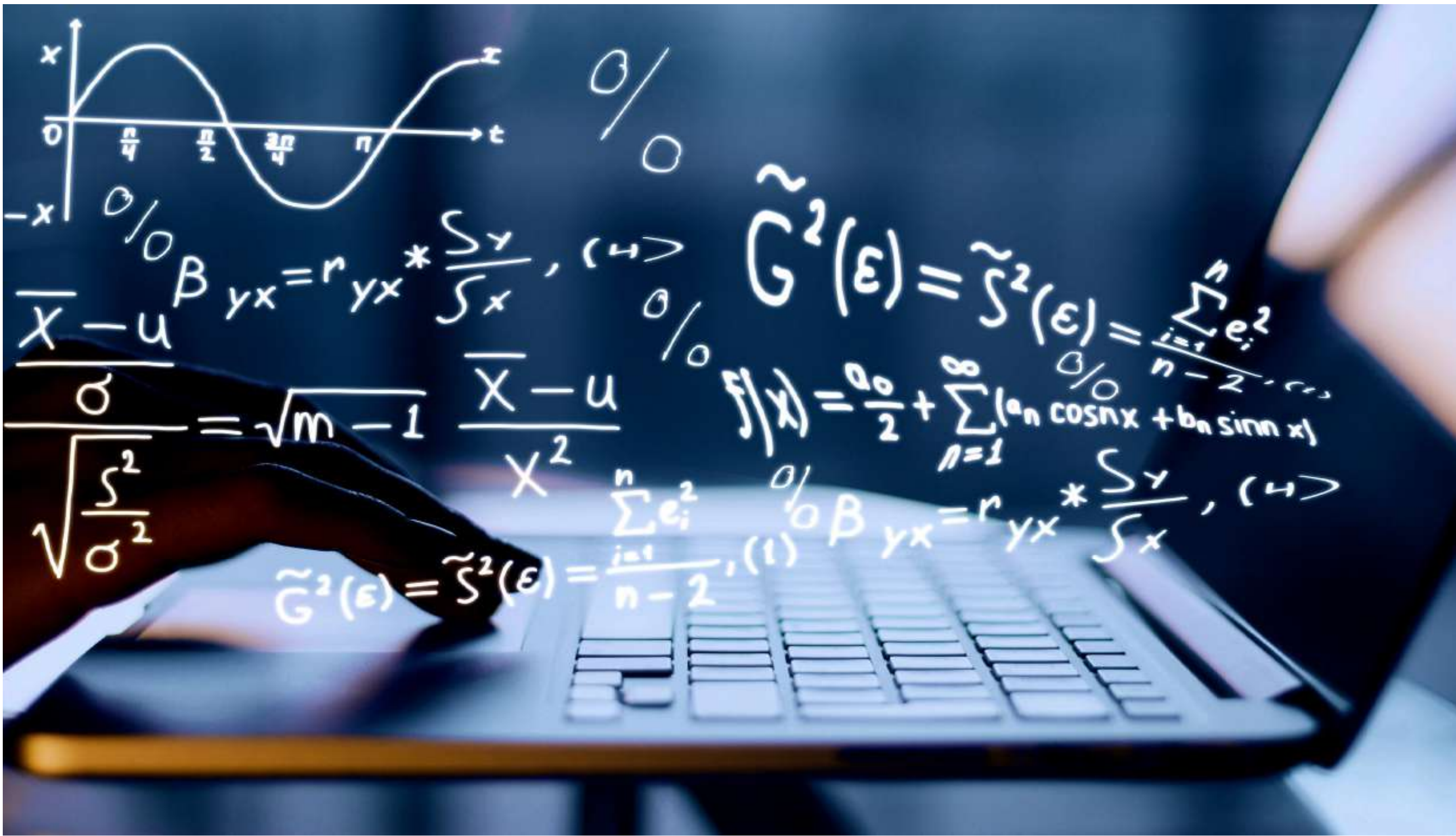
Trending Research in Pure and Applied Mathematics

Dr. Parama Dutta



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By:

Dr. Parama Dutta
Assistant Professor
Department of Mathematics
Lakhimpur Girls' College
Khelmati, Lakhimpur, Assam

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Preface

The present book is a collection of several survey articles written by young Mathematics researchers from Assam, describing mainly their work. The topics selected are from various areas of Mathematics. The book begins with an article in Number Theory enunciating the recent developments in the area. Then we have several articles in Differential Equations, Computational Fluid Dynamics, Algebra and Topology.

It is hoped that this book would serve as a ready reference for someone who is interested in the topics presented here. A generous sprinkling of open problems in almost all the articles makes it easy to look for research problems in these areas and the editor hopes that it will serve the mathematical community well.

Lakhimpur
October 2020

Dr. Parama Dutta
Assistant Professor
Lakhimpur Girls' College

Contents

Relations between the number of representation of n in two certain quadratic forms

Hirakjyoti Das
Department of Mathematical Sciences, Tezpur University, Sonitpur, Assam, India

email: hirakjyoti@live.in

Abstract. In this review article, we provide the compendium of results on the relations between the number of representation of a positive integer in the quadratic forms $c_1x_1^2 + c_2x_2^2 + \cdots + c_kx_k^2$ and $c_1x_1(x_1+1)/2 + c_2x_2(x_2+1)/2 + \cdots + c_kx_k(x_k+1)/2$.

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1 Introduction

For positive integers c_1, c_2, c_3, c_4, and n, let

$$r(c_1, c_2, c_3, c_4; n) := \text{card}\{(x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 \mid c_1x_1^2 + c_2x_2^2 + c_3x_3^2 + c_4x_4^2 = n\},$$

$$t(c_1, c_2, c_3, c_4; n) := \text{card}\Big\{(x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 \mid c_1\frac{x_1(x_1+1)}{2} + c_2\frac{x_2(x_2+1)}{2} + c_3\frac{x_3(x_3+1)}{2} + c_4\frac{x_4(x_4+1)}{2} = n\Big\}.$$

The generating functions of $r(c_1,c_2,c_3,c_4;n)$ and $t(c_1,c_2,c_3,c_4;n)$ are given by

$$\sum_{n=0}^{\infty} r(c_1,c_2,c_3,c_4;n)q^n = \varphi\left(q^{c_1}\right)\varphi\left(q^{c_2}\right)\varphi\left(q^{c_3}\right)\varphi\left(q^{c_4}\right),$$
$$\sum_{n=0}^{\infty} t(c_1,c_2,c_3,c_4;n)q^n = \psi\left(q^{c_1}\right)\psi\left(q^{c_2}\right)\psi\left(q^{c_3}\right)\psi\left(q^{c_4}\right),$$

where $\varphi(q)$ and $\psi(q)$ are Ramanujan's theta functions. Jacobi, in 1828 proved that

$$r(1,1,1,1;n) = \sum_{d|n,4\nmid d} d.$$

There have been extensive work the exact formulas for $r(c_1,c_2,c_3,c_4;n)$ and $t(c_1,c_2,c_3,c_4;n)$ which involve the divisor function $\sigma(n)$ which is defined as the sum of the positive divisors of n. In recent years, many researchers have found relations between $r(c_1,c_2,c_3,c_4;n)$ and $t(c_1,c_2,c_3,c_4;n)$. In this review article, we collect and state all such relations available in the literature.

2 Relations between $r(c_1,c_2,c_3,c_4;n)$ and $t(c_1,c_2,c_3,c_4;n)$

In 2016, Sun [2] found the following results.

Theorem 2.1. (Z. -H. Sun, [2]) *Let* $a \in \{1,3,5,\ldots\}$ *and* $m \in \{0,1,2,\ldots\}$. *For* $n \in \mathbb{N}$, *we have*

$$\begin{aligned}
&t(a,a,2a,8m+4;n)\\
&= \frac{2}{3}\left(r(a,a,a,4m+2;4n+4m+2a+2) - r(a,a,a,4m+2;n+m+(a+1)/2)\right)\\
&= \frac{2}{3}\left(r(a,a,2a,8m+4;8n+8m+4a+4) - r(a,a,2a,8m+4;2n+2m+a+1)\right).
\end{aligned}$$

Theorem 2.2. (Z. -H. Sun, [2]) *Let* $a \in \{1,3,5,\ldots\}$ *and* $k,m \in \{0,1,2,\ldots\}$ *and* $k \equiv m \pmod 2$. *For* $n \in \mathbb{N}$, *we have*

$$\begin{aligned}
&t(a,3a,4k+2,4m+2;n)\\
&= \frac{2}{3}(r(a,3a,4k+2,4m+2;8n+4m+4k+4a+4)\\
&\quad - r(a,3a,4k+2,4m+2;2n+m+k+a+1)).
\end{aligned}$$

Theorem 2.3. (Z. -H. Sun, [2]) *Let $a, k \in \mathbb{N}$ with $2 \nmid ak$. For $n \in \mathbb{N}$, we have*

$$t(a,3a,k,k;m) = \frac{2}{3}r(a,3a,2k,2k;8m+4a+2k).$$

Theorem 2.4. (Z. -H. Sun, [2]) *Let $a \in \{1,3,5,\ldots\}$, $k,m \in \{0,1,2,\ldots\}$ and $n \in \mathbb{N}$. If $n \equiv k+\frac{a-1}{2} \pmod 2$, then*

$$t(a,3a,8k+4,4m+2;n) = \frac{2}{3}r(a,3a,8k+4,4m+2;8n+4m+8k+4a+6).$$

Theorem 2.5. (Z. -H. Sun, [2]) *If $n \in \mathbb{N}$ and $8n+13 = 3^{\beta}n_1$ with $n_1 \in \mathbb{N}$ and $3 \nmid n_1$, then*

$$t(1,3,3,6;n) = \frac{2}{5}r(1,3,3,6;8n+13).$$

Theorem 2.6. (Z. -H. Sun, [2]) *If $n \in \mathbb{N}$ and $n \equiv 1,2 \pmod 4$, then*

$$t(1,1,4,6;n) = 2r(1,1,4,6;2n+3).$$

Theorem 2.7. (Z. -H. Sun, [2]) *If $n \in \mathbb{N}$ and $n \equiv 1 \pmod 4$, then*

$$t(2,2,3,9;n) = \frac{4}{3}r(2,2,3,9;2n+4).$$

Theorem 2.8. (Z. -H. Sun, [2]) *For $n \in \mathbb{N}$, we have*

$$t(1,2,2,6;n) = \frac{1}{2}r(1,1,4,6;8n+11)$$
$$t(1,1,8,12;2n) = \frac{1}{2}r(1,1,8,12;16n+22).$$

Besides these above results, Sun conjectured several relations between $r(c_1,c_2,c_3,c_4;n)$ and $t(c_1,c_2,c_3,c_4;n)$. The conjectures are as follows.

Conjecture 2.9. *Let $n \in \mathbb{N}$ with $n \equiv 0,3 \pmod 4$. Then*

$$t(1,1,4,6;n) = \frac{2}{3}r(1,1,4,6;8n+12) - r(1,1,4,6;2n+3).$$

Conjecture 2.10. *Let $n \in \mathbb{N}$. If $3 \mid n$, then*

$$t(1,1,8,12;n) = \frac{1}{2} r(1,1,8,12;8n+22).$$

Conjecture 2.11. *Let $m \in \mathbb{N}$. Then*

$$t(1,3,8,8;3m) = \frac{1}{3} r(1,3,8,8;24m+20) - 2r(1,3,8,8;6m+5).$$

Conjecture 2.12. *Let $n \in \mathbb{N}$ with $n \equiv 0 \pmod 6$. Then*

$$t(1,2,3,8;n) = \frac{2}{3} r(1,2,3,8;8n+14) - 2r(1,2,3,8;4n+7).$$

Conjecture 2.13. *Let $n \in \mathbb{N}$ with $n \equiv 0,2 \pmod 8$. Then*

$$t(1,2,4,17;n) = 4r(1,2,4,17;n+3).$$

Conjecture 2.14. *Let $n \in \mathbb{N}$. If $n \equiv 2,3 \pmod 5$, then*

$$t(1,1,5,8;n) = \frac{1}{2} r(1,1,5,8;8n+15).$$

Conjecture 2.15. *Let $n \in \mathbb{N}$. If $n \equiv 0,3,4,6,7 \pmod 9$, then*

$$t(1,1,8,9;n) = \frac{1}{2} r(1,1,8,9;8n+19).$$

Conjecture 2.16. *Let $n \in \mathbb{N}$. If $n \equiv 0,4,7,8,9,10 \pmod{13}$, then*

$$t(1,1,8,13;n) = \frac{1}{2} r(1,1,8,13;8n+23).$$

Conjecture 2.17. *Let $n \in \mathbb{N}$. If $n \equiv 0,3,5,6,7 \pmod{11}$, then*

$$t(1,1,4,11;n) = \frac{1}{3} r(1,1,4,11;8n+17).$$

Conjecture 2.18. *Let $n \in \mathbb{N}$. If $n \equiv 0,1,2,4,7 \pmod{11}$, then*

$$t(1,1,2,22;n) = \frac{1}{3} r(1,1,2,22;8n+26).$$

Conjecture 2.19. *Let $n \in \mathbb{N}$ with $n \equiv 1 \pmod 3$. Then*

$$t(1,3,12,36;n) = \frac{1}{2}r(1,3,12,36;8n+52) - 2r(1,3,12,36;2n+13).$$

Conjecture 2.20. *Let $n \in \mathbb{N}$ with $n \equiv 1 \pmod 4$. Then*

$$t(3,5,20,32;n) = \frac{1}{2}r(3,5,20,32;8n+60) - 2r(3,5,20,32;2n+15).$$

Conjecture 2.21. *Let $n \in \mathbb{N}$ with $n \equiv 1 \pmod 4$. Then*

$$t(1,6,15,18;n) = \frac{2}{3}r(1,6,15,18;8n+40) - 2r(1,6,15,18;2n+10).$$

Conjecture 2.22. *Let $n \in \mathbb{N}$ with $n \equiv 1 \pmod 3$. Then*

$$t(1,6,18,27;n) = \frac{2}{3}r(1,6,18,27;8n+52) - 2r(1,6,18,27;2n+13).$$

Conjecture 2.23. *Let $n \in \mathbb{N}$ with $n \equiv 1 \pmod 3$. Then*

$$t(1,8,9,18;n) = \frac{2}{3}r(1,8,9,18;8n+36) - 2r(1,8,9,18;2n+9).$$

Conjecture 2.24. *Let $n \in \mathbb{N}$ with $4 \mid n$. Then*

$$t(1,7,10,30;n) = \frac{2}{3}r(1,7,10,30;8n+48) - 2r(1,7,10,30;2n+12).$$

Conjecture 2.25. *Let $n \in \mathbb{N}$ with $n \equiv 3 \pmod 4$. Then*

$$t(1,10,15,30;n) = \frac{2}{3}r(1,10,15,30;8n+56) - 2r(1,10,15,30;2n+14).$$

Conjecture 2.26. *Let $n \in \mathbb{N}$ with $n \equiv 2 \pmod 8$. Then*

$$t(1,7,28,28;n) = \frac{2}{3}r(1,7,28,28;8n+64) - 2r(1,7,28,28;2n+16).$$

Conjecture 2.27. *Let $n \in \mathbb{N}$ with $n \equiv 8 \pmod 9$. Then*

$$t(1,9,16,18;n) = \frac{2}{3}r(1,9,16,18;8n+44) - 2r(1,9,16,18;2n+11).$$

Conjecture 2.28. *Let $n \in \mathbb{N}$ with $n \equiv 1,7 \pmod 9$. Then*

$$t(1,9,18,24;n) = \frac{2}{3}r(1,9,18,24;8n+52) - 2r(1,9,18,24;2n+13).$$

Conjecture 2.29. *Let $n \in \mathbb{N}$ with $n \equiv 1,4 \pmod 9$. Then*

$$t(1,9,18,32;n) = \frac{2}{3}r(1,9,18,32;8n+60) - 2r(1,9,18,32;2n+15).$$

Conjecture 2.30. *Let $n \in \mathbb{N}$ with $n \equiv 5 \pmod 9$. Then*

$$t(1,9,18,40;n) = \frac{2}{3}r(1,9,18,40;8n+68) - 2r(1,9,18,40;2n+17).$$

Conjecture 2.31. *Let $n \in \mathbb{N}$ with $n \equiv 2,5 \pmod 9$. Then*

$$t(1,10,27,30;n) = \frac{2}{3}r(1,10,27,30;8n+68) - 2r(1,10,27,30;2n+17).$$

Some of the conjectures have been confirmed by the authors in [1, 6, 7, 8].

In 2017, Wang and Sun [5] found the following theorems.

Theorem 2.32. (Wang-Sun, [5]) *Let $m, n \in \mathbb{N}$ and $a \in \{1,3,5,\dots\}$, then*

$$t(a,a,2a,8m;n) = \frac{2}{3}r(a,a,2a,8m;8n+8m+4a) - 2r(a,a,2a,8m;2n+2m+a).$$

Theorem 2.33. (Wang-Sun, [5]) *Let $a \in \{1,3,5,\dots\}$ and $k, m \in \{0,1,2,\dots\}$. For $n \in \mathbb{N}$, we have*

$$\begin{aligned} t(a,3a,8k+2,8m+6;n) = \frac{2}{3}&r(a,3a,8k+2,8m+6;8n+8k+8m+4a+8) \\ &- 2r(a,3a,8k+2,8m+6;2n+2k+2m+a+2). \end{aligned}$$

Theorem 2.34. (Wang-Sun, [5]) *Let $a \in \{1,3,5,\dots\}$, $m \in \{0,1,2,\dots\}$ and $n \in \mathbb{N}$. If $n \equiv m + \frac{a-1}{2} \pmod 2$, then*

$$\begin{aligned} t(a,3a,8m+4,8m+4;n) = \frac{2}{3}&r(a,3a,8m+4,8m+4;8n+16m+4a+8) \\ &- 2r(a,3a,8m+4,8m+4;2n+4m+a+2). \end{aligned}$$

Theorem 2.35. (Wang-Sun, [5]) *Let* $a \in \{1,3,5,\ldots\}$, $k,m \in \{0,1,2,\ldots\}$ *and* $n \in \mathbb{N}$. *If* $n \equiv \frac{a-1}{2} \pmod 2$, *then*

$$\begin{aligned} t(a,3a,16k+4,16m+4;n) =& \frac{2}{3} r(a,3a,16k+4,16m+4;8n+16k+16m+4a+8) \\ & -2r(a,3a,16k+4,16m+4;2n+4k+4m+a+2). \end{aligned}$$

Recently, in 2019, Sun [3] again found many relations which are stated below.

Theorem 2.36. (Sun, [3]) *Suppose* $a,b,c,d,n \in \mathbb{N}, 2 \nmid abc$ *and* $a \equiv b \equiv c \pmod 4$. *Then*

$$\begin{aligned} & t(a,b,c,d;n) \\ & = r(a,b,c,d;8n+a+b+c+d) - r(a,b,c,4d;8n+a+b+c+d). \end{aligned}$$

Corollary 2.37. (Sun, [3]) *Let* $a,b,c,d,n \in \mathbb{N}$ *with* $a \equiv b \equiv c \equiv \pm 1 \pmod 4$ *and* $d \equiv 4 \pmod 8$. *Then*

$$t(a,b,c,d;n) = r(a,b,c,d;8n+a+b+c+d).$$

Theorem 2.38. (Sun, [3]) *Suppose* $a,b,c,d,n \in \mathbb{N}$, $2 \nmid abcd$ *and* $a \equiv b \equiv c \equiv d \pmod 4$. *Then*

$$t(a,b,c,d;n) = r\Big(a,b,c,d;8n+a+b+c+d\Big) - r\left(a,b,c,d;2n+\frac{a+b+c+d}{2}\right).$$

Theorem 2.39. (Sun, [3]) *Suppose* $a,b,c,d,n \in \mathbb{N}, 2 \nmid a, 2 \mid b, 2 \mid c, 8 \nmid b, 8 \nmid c$ *and* $8 \nmid b+c$. *Then*

$$\begin{aligned} & t(a,b,c,d;n) \\ & = r(a,b,c,d;8n+a+b+c+d) - r(a,b,c,4d;8n+a+b+c+d). \end{aligned}$$

Theorem 2.40. (Sun, [3]) *Suppose* $a,c,d,n \in \mathbb{N}$, $2 \nmid a$ *and* $4 \nmid c$. *Then*

$$t(a,3a,c,d;n) = 2r(4a,12a,c,d;8n+4a+c+d) - 2r(4a,12a,c,4d;8n+4a+c+d).$$

Corollary 2.41. (Sun, [3]) *Suppose* $a,c,d,n \in \mathbb{N}$, $2 \nmid ac$ *and* $d \equiv 2, c \pmod 4$. *Then*

$$t(a,3a,c,d;n) = 2r(4a,12a,c,d;8n+4a+c+d).$$

Theorem 2.42. (Sun, [3]) *Let $a,b,d,n \in \mathbb{N}$ with $2 \nmid ab$. Then*

$$t(a,3a,2b,d;n) = \frac{2}{3}\big(r(a,3a,2b,d;8n+4a+2b+d) - r(a,3a,2b,4d;8n+4a+2b+d)\big).$$

Theorem 2.43. (Sun, [3]) *Let $a,d,n \in \mathbb{N}$. Then*

$$t(a,3a,9a,d;n) = \frac{1}{2}\big(r(a,3a,9a,d;8n+13a+d) - r(a,3a,9a,4d;8n+13a+d)\big).$$

Theorem 2.44. (Sun, [3]) *Let $a,b,c,n \in \mathbb{N}$ with $2 \nmid ab$ and $n \not\equiv \frac{a+b}{2} \pmod 2$. Then*

$$\begin{aligned}&t(a,3a,4b,2c;n)\\&= \frac{2}{3}\big(r(a,3a,4b,2c;8n+4a+4b+2c) - r(a,3a,4b,8c;8n+4a+4b+2c)\big).\end{aligned}$$

Theorem 2.45. (Sun, [3]) *Let $m,n \in \mathbb{N}$.*

(i) *If there is a prime divisor p of $2m+1$ such that $(\frac{8n+5}{p}) = -1$, then*

$$t(1,2,2,4m+2;n) = \frac{1}{2}r(1,1,4,4m+2;8n+4m+7).$$

(ii) *If there is a prime divisor p of $2m+1$ such that $(\frac{8n+9}{p}) = -1$, then*

$$t(1,4,4,4m+2;n) = \frac{1}{4}r(1,1,4,4m+2;8n+4m+11).$$

Theorem 2.46. (Sun, [3]) *Let $a,b \in \{1,3,5,\dots\}$. Then for $n \in \mathbb{N}$,*

$$t(a,a,2b,4b;n) = r(a,a,b,2b;4n+a+3b) - r(a,a,b,2b;2n+(a+3b)/2).$$

Theorem 2.47. (Sun, [3]) *Let $a,b \in \{1,3,5,\dots\}$ and $n \in \mathbb{N}$. Then*

$$t(a,2a,4a,b;n) = \frac{1}{4}\Big(r(a,a,a,2b;16n+14a+2b) - r(a,a,2a,b;8n+7a+b)\Big).$$

Theorem 2.48. (Sun, [3]) *Let $a,b \in \{1,3,5,\dots\}$ and $n \in \mathbb{N}$. Then*

$$r(a,a,2a,b;2n+a+b) = \frac{1}{3}\Big(r(a,a,a,2b;4n+2a+2b) + 2r\big(a,a,a,2b;n+\frac{a+b}{2}\big)\Big)$$
$$t(a,2a,4a,b;n) = \frac{1}{6}\Big(r(a,a,a,2b;16n+14a+2b) - r\big(a,a,a,2b;4n+\frac{7a+b}{2}\big)\Big).$$

Theorem 2.49. (Sun, [3]) *Let* $a, b \in \{1,3,5,\ldots\}$ *and* $n \in \mathbb{N}$. *Then*

$$t(a,a,6a,b;n) = \frac{1}{2}\Big(r(a,a,3a,2b;16n+16a+2b) - r(a,a,6a,b;8n+8a+b)\Big).$$

Theorem 2.50. (Sun, [3]) *Suppose* $a, b, n \in \mathbb{N}$, $2 \nmid a$ *and* $b \equiv 2 \pmod 4$. *Then*

$$t(a,a,b,b;n) = r(a,a,b,b;4n+a+b).$$

Theorem 2.51. (Sun, [3]) *Let* $a, b \in \{1,3,5,\ldots\}$ *and* $n \in \mathbb{N}$. *Then*

$$t(a,a,b,b;n) = r(a,a,b,b;4n+a+b) - r(a,a,b,b;2n+(a+b)/2).$$

Theorem 2.52. (Sun, [3]) *Let* $a, b, n \in \mathbb{N}$, $2 \nmid ab$ *and* $4 \mid a-b$. *Then*

$$t(a,2a,b,2b;n) = r(a,2a,b,2b;8n+3(a+b)) - r(a,2a,b,2b;4n+3(a+b)/2\Big).$$

Theorem 2.53. (Sun, [3]) *Let* $n \in \mathbb{N}$. *Then*

$$t(1,1,1,6;n) = \frac{1}{6}\Big(r(1,1,1,6;32n+36) - r(1,1,1,6;8n+9)\Big).$$

Theorem 2.54. (Sun, [3]) *Let* $n \in \mathbb{N}$. *Then*

$$t(1,1,1,7;n) = 4r(1,1,1,7;4n+5) - 2r(1,1,1,7;8n+10),$$

$$t(1,7,7,7;n) = 4r(1,7,7,7;4n+11) - 2r(1,7,7,7;8n+22).$$

Theorem 2.55. (Sun, [3]) *Let* $n \in \mathbb{N}$. *Then*

$$t(1,2,6,6;n) = 2r(1,2,6,6;8n+15) - r(1,2,6,6;16n+30),$$

$$t(2,2,3,6;n) = 2r(2,2,3,6;8n+13) - r(2,2,3,6;16n+26).$$

Sun [3] also left some relations of $t(a;b;c;d;n)$ and $r(a;b;c;d;n)$ as conjectures. Interested readers may go through the conjectures in [3].

Again in 2019, Sun [4] has come up with several relations of $t(a;b;c;d;n)$ and $r(a;b;c;d;n)$. We state those results in the next.

Theorem 2.56. (Sun, [4]) *Let $a, b \in \mathbb{Z}^+$ with $2 \nmid ab$. For $n = 0, 1, 2, \ldots$ we have*

$$t(a, 2a, 2a, 2b; n) = \frac{1}{2} r(a, a, 4a, 2b; 8n + 5a + 2b).$$

Theorem 2.57. (Sun, [4]) *Let $a, b \in \mathbb{Z}^+$ with $ab \equiv -1 \pmod 4$. For $n \in \mathbb{Z}^+$ we have*

$$t(a, a, 2a, b; n) = 2r(a, 4a, 8a, b; 8n + 4a + b).$$

Theorem 2.58. (Sun, [4]) *Let $a, b \in \mathbb{Z}^+$ with $2 \nmid ab$. For $n \in \mathbb{Z}^+$ we have*

$$t(a, 3a, 3a, 2b; n) = r(3a, 3a, 4a, 2b; 8n + 7a + 2b).$$

Theorem 2.59. (Sun, [4]) *Let $a, b, n \in \mathbb{Z}^+$ with $2 \nmid ab$. For $n \in \mathbb{Z}^+$ with $n \equiv \frac{a-b}{2} \pmod 2$ we have*

$$t(a, 3a, b, 3b; n) = 4r(a, 3a, b, 3b; 2n + a + b).$$

Theorem 2.60. (Sun, [4]) *Let $a, b \in \mathbb{Z}^+$ with $ab \equiv -1 \pmod 4$. For $n \in \mathbb{Z}^+$ we have*

$$t(2a, 2a, 3a, b; n) = \frac{1}{3} r(a, 3a, 16a, 4b; 32n + 28a + 4b).$$

Theorem 2.61. (Sun, [4]) *Let $a, b \in \mathbb{Z}^+$ with $ab \equiv 1 \pmod 4$. For $n \in \mathbb{Z}^+$ we have*

$$t(a, 6a, 6a, b; n) = \frac{1}{3} r(a, 3a, 48a, 4b; 32n + 52a + 4b).$$

Theorem 2.62. (Sun, [4]) *Suppose $a, b, n \in \mathbb{Z}^+$, $(a, b) = 1$ and there is an odd prime divisor p of b such that $(\frac{a(8n+9a)}{p}) = -1$. Then*

$$t(a, 4a, 4a, b; n) = \frac{1}{2}(r(a, 4a, 4a, b; 8n + 9a + b) - r(a, 4a, 4a, 4b; 8n + 9a + b)).$$

Theorem 2.63. (Sun, [4]) *Suppose $a, b, n \in \mathbb{Z}^+$, $(a, b) = 1$, $b \not\equiv 0, -a \pmod 4$ and there is an odd prime divisor p of b such that $(\frac{a(8n+9a)}{p}) = -1$. Then*

$$t(a, 4a, 4a, b; n) = \frac{1}{2} r(a, 4a, 4a, b; 8n + 9a + b).$$

Corollary 2.64. (Sun, [4]) *Suppose $a, b, n \in \mathbb{Z}^+$, $3 \nmid a$, $3 \mid b$, $b \not\equiv 0, 3a \pmod 4$ and $3 \mid n - a$.*

Then

$$t(a,4a,4a,b;n) = \frac{1}{2}r(a,4a,4a,b;8n+9a+b).$$

Corollary 2.65. (Sun, [4]) *Suppose* $m,n \in \mathbb{Z}^+$, $m \equiv 1,2 \pmod 4$ *and* $n \equiv 1,3 \pmod 5$. *Then*

$$t(1,4,4,5m;n) = \frac{1}{2}r(1,4,4,5m;8n+5m+9).$$

Theorem 2.66. (Sun, [4]) *Suppose* $a,b,n \in \mathbb{Z}^+$, $(a,b)=1$ *and there is an odd prime divisor* p *of* b *such that* $(\frac{a(4n+5a)}{p}) = -1$. *Then*

$$t(a,a,8a,b;n) = \frac{1}{2}(r(a,a,8a,b;8n+10a+b) - r(a,a,8a,4b;8n+10a+b)).$$

Theorem 2.67. (Sun, [4]) *Suppose* $a,b,n \in \mathbb{Z}^+$, $(a,b)=1$ *and there is an odd prime divisor* p *of* b *such that* $(\frac{a(4n+5a)}{p}) = -1$. *Assume that* a *is even or* $ab \equiv 1,4,5 \pmod 8$ *for odd* a. *Then*

$$t(a,a,8a,b;n) = \frac{1}{2}r(a,a,8a,b;8n+10a+b).$$

Theorem 2.68. (Sun, [4]) *Let* $n \in \mathbb{Z}^+$ *Then*

$$\begin{aligned}
&t(1,4,7,8;n) = 2r(1,4,7,8;2n+5) \quad \textit{for} \quad n \equiv 3 \pmod 4,\\
&t(1,4,8,15;n) = 2r(1,4,8,15;2n+7) \quad \textit{for} \quad n \equiv 2 \pmod 4,\\
&t(3,5,12,24;n) = 2r(3,5,12,24;2n+11) \quad \textit{for} \quad n \equiv 3 \pmod 4,\\
&t(3,5,20,40;n) = 2r(3,5,20,40;2n+17) \quad \textit{for} \quad n \equiv 3 \pmod 4.
\end{aligned}$$

Theorem 2.69. (Sun, [4]) *For* $n \in \mathbb{Z}^+$ *we have*

$$\begin{aligned}
&t(2,3,3,4;n) = 2r(2,3,3,4;2n+3) \quad \textit{for} \quad n \equiv 2,3 \pmod 4,\\
&t(2,3,3,12;n) = 2r(2,3,3,12;2n+5) \quad \textit{for} \quad n \equiv 0,1 \pmod 4,\\
&t(2,3,3,24;n) = 4r(2,3,3,24;2n+8) \quad \textit{for} \quad n \equiv 2 \pmod 4,\\
&t(2,3,3,36;n) = 2r(2,3,3,36;2n+11) \quad \textit{for} \quad n \equiv 2,3 \pmod 4,\\
&t(1,1,6,12;n) = 2r(1,1,6,12;2n+5) \quad \textit{for} \quad n \equiv 0,3 \pmod 4,\\
&t(1,1,6,16;n) = \begin{cases} r(1,1,3,8;n+3) & \textit{if} \quad n \equiv 2 \pmod 8,\\ 4r(1,1,3,8;n+3) & \textit{if} \quad n \equiv 4 \pmod 8.\end{cases}
\end{aligned}$$

Theorem 2.70. (Sun, [4]) *For $n \in \mathbb{Z}^+$ with $n \equiv 3,5 \pmod 8$,*

$$t(1,1,2,12;n) = 4r(1,1,4,6;n+2) = \frac{8}{3}r(1,1,1,6;n+2)$$
$$t(3,3,4,6;n) = \frac{8}{3}r(2,3,3,3;n+2).$$

Using Sun's [4] technique one can find several similar relations like

$$t(3,3,4,18;n) = 2r(3,3,4,18;2n+7) \quad \text{for} \quad n \equiv 0,1 \pmod 4,$$
$$t(1,3,8,12;n) = 4r(1,3,8,12;n+3) \quad \text{for} \quad n \equiv 2,4 \pmod 8,$$
$$t(1,1,2,28;n) = 4r(1,1,2,28;n+4) \quad \text{for} \quad n \equiv 1,3 \pmod 8,$$
$$t(1,3,4,24;n) = 4r(1,3,4,24;n+4) \quad \text{for} \quad n \equiv 1,3 \pmod 8,$$
$$t(2,3,3,48;n) = r(2,3,3,48;2n+14) \quad \text{for} \quad n \equiv 0 \pmod 8,$$
$$t(1,1,8,14;n) = 8r(1,1,8,14;n+3) \quad \text{for} \quad n \equiv 1 \pmod 8,$$
$$t(1,1,10,20;n) = 4r(1,1,10,20;n+4) \quad \text{for} \quad n \equiv 1 \pmod 8,$$
$$t(1,1,14,16;n) = 4r(1,1,14,16;n+4) \quad \text{for} \quad n \equiv 1 \pmod 8,$$
$$t(1,2,7,14;n) = 8r(1,2,7,14;n+3) \quad \text{for} \quad n \equiv 1 \pmod 8,$$
$$t(1,1,8,30;n) = 4r(1,1,8,30;2n+10) \quad \text{for} \quad n \equiv 1 \pmod 8,$$
$$t(1,3,4,16;n) = \frac{4}{3}r(1,3,4,16;2n+6) \quad \text{for} \quad n \equiv 1 \pmod 8,$$
$$t(3,3,10,48;n) = 4r(3,3,10,48;2n+16) \quad \text{for} \quad n \equiv 1 \pmod 8,$$
$$t(1,1,8,14;n) = 4r(1,1,8,14;2n+6) \quad \text{for} \quad n \equiv 3 \pmod 8,$$
$$t(2,15,15,24;n) = 4r(2,15,15,24;2n+14) \quad \text{for} \quad n \equiv 3 \pmod 8,$$
$$t(5,5,6,8;n) = 4r(5,5,6,8;2n+6) \quad \text{for} \quad n \equiv 3 \pmod 8,$$
$$t(1,3,12,48;n) = \frac{4}{3}r(1,3,12,48;2n+16) \quad \text{for} \quad n \equiv 4 \pmod 8,$$
$$t(2,4,5,5;n) = 4r(2,4,5,5;n+2) \quad \text{for} \quad n \equiv 5 \pmod 8,$$
$$t(4,7,7,14;n) = 4r(4,7,7,14;n+4) \quad \text{for} \quad n \equiv 5 \pmod 8,$$
$$t(1,1,16,30;n) = 4r(1,1,16,30;2n+12) \quad \text{for} \quad n \equiv 5 \pmod 8,$$
$$t(1,1,30,40;n) = 4r(1,1,30,40;2n+18) \quad \text{for} \quad n \equiv 5 \pmod 8,$$
$$t(1,3,16,36;n) = \frac{4}{3}r(1,3,16,36;2n+14) \quad \text{for} \quad n \equiv 5 \pmod 8,$$

$$t(2,3,3,32;n) = 4r(2,3,3,32;2n+10) \quad \text{for} \quad n \equiv 5 \pmod 8,$$
$$t(2,7,7,24;n) = 4r(2,7,7,24;2n+10) \quad \text{for} \quad n \equiv 5 \pmod 8,$$
$$t(3,3,10,24;n) = 4r(3,3,10,24;2n+10) \quad \text{for} \quad n \equiv 5 \pmod 8,$$
$$t(1,7,16,16;n) = 4r(1,7,16,16;n+5) \quad \text{for} \quad n \equiv 6 \pmod 8,$$
$$t(2,3,3,48;n) = 4r(2,3,3,48;2n+14) \quad \text{for} \quad n \equiv 6 \pmod 8,$$
$$t(1,1,10,20;n) = 4r(1,1,10,20;n+4) \quad \text{for} \quad n \equiv 7 \pmod 8,$$
$$t(2,4,5,5;n) = 4r(2,4,5,5;n+2) \quad \text{for} \quad n \equiv 7 \pmod 8,$$
$$t(4,7,7,14;n) = 4r(4,7,7,14;n+4) \quad \text{for} \quad n \equiv 7 \pmod 8,$$
$$t(1,1,14,16;n) = 4r(1,1,14,16;2n+8) \quad \text{for} \quad n \equiv 7 \pmod 8,$$
$$t(5,5,6,40;n) = 4r(5,5,6,40;2n+14) \quad \text{for} \quad n \equiv 7 \pmod 8.$$

3 Conclusion

There are still active conjectures on the relations between $t(a;b;c;d;n)$ and $r(a;b;c;d;n)$. Many active researchers are also working in this field to prove new relations.

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The role of compactness in analysis and various compactifications

Ankur Sharmah
Department of Mathematical Sciences, Tezpur University, Sonitpur, Assam, India

email: ankursarmah189@gmail.com

Abstract. Here we present a short survey on role of compactness in analysis and topology. Further different compatification mechanisms and their applications

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1 Introduction

The notions metric space, topological space are defined as a set endowed with a structure designed for several generalisation. As people proceeded in its journey of research and development of mathematics, it turned out that some sets exhibit properties similar to finite sets. In the categories of topological spaces and metric spaces, these almost finite objects are termed as Compact Spaces. After decades of working with different topological spaces mathematical world has realized that great proportions of mathematical theories turn out to be following.

1. Trivial for finite sets.

2. True and reasonably simple for infinite compact spaces.

3. Either false or extremely difficult to prove for non compact spaces.

Or, it is not always handy to work with non compact spaces. After years of studying a non compact topological space X, it is realised that for non compact spaces it is often convenient to construct a space which contains X as a subspace and itself compact. Question arises how it will work. So, Realizing a space X is a subspace of Y (compact), properties of larger space Y often gives new insight into those of X itself. The more we understand a space more it will help for further study. Compactification in general topology mainly deals with construction of specific Y's. for example we can observe the real line $\mathbb{R}$ is not compact because it contains sequences such as $1, 2, 3, \dots$ which are trying to escape the real line, and are not leaving behind any convergent subsequences. However, one can often recover compactness by adding a few more points to the space. For instance, one can compactify the real line by adding one point at either end of the real line, $+\infty$ and $-\infty$. The resulting object, known as the extended real line $[-\infty, +\infty]$, can be given a topology. The extended real line is compact: any sequence $\{x_n\}$ of extended real numbers will have a subsequence that either converges to $+\infty$, converges to $-\infty$, or converges to a finite number.

Compactification is a way to take a space X that is not compact, and to add a little bit to it in such a way that we produce a new compact space. To be more precise, we want to construct a compact space $\overline{X}$ such that X is homeomorphic to an open dense subset of $\overline{X}$. We want it to be dense because we want to alter X as little as possible, which means we don't want to have to add very much to X in order to get to $\overline{X}$. (Here required compact space is written as $\overline{X}$ to be observable as X is dense in its compactification).

2 Some Prerequisites

In this chapter, the basic results which will be used for the study of Compactification directly or indirectly are mentioned in briefly.

Heine-Borel Theorem: Every open cover of a closed and bounded subset of $\mathbb{R}$ (space of real numbers) has a finite subcover.

Tychonoff Theorem: Arbitrary product of compact spaces are compact.
Indeed Heine-Borel theorem and Tychonoffs theorem set the definition for a Topological space having similar properties those of closed and bounded subset of $\mathbb{R}$ so called compact spaces, defined as following :

2.1 Compact Space

X, a topological space is said to be a compact space if and only if every open cover of X has a finite subcover.
For example, (X, T) be a topological space where X is a set and $T = \{\phi, X\}$ is the topology. Then trivially, X is a compact space.
Again, for a, b in $\mathbb{R}$, $[a, b]$ being a closed and bounded subset of $\mathbb{R}$ is compact.
Identifying Compact Spaces

- Every closed subspace of a compact space is compact.
- Every compact subspace of a Hausdorff space (The space in which two different points can be separated by two disjoint open sets) is closed.
- A continuous image of a compact space is a compact space.
- Arbitrary Product space of compact spaces (product topology) are compact. (by Tychonoff theorem).
- A subspace A of $\mathbb{R}^n$ is compact if it is closed and bounded in the euclidean metric or the square metric.

2.2 Separation Properties

X be a topological space. We can separate two subsets of X by following :
1. Separation by open sets.
2. Separation by continuous mapping.

2.2.1 Separation Axioms

The separation axiom are listed below.
T_0 **space**: each pair of distinct points x and y either there exist an open set containing

x not containing y or there exist an open set containing y not containing x.
T_1 Space: each pair of distinct points x and y either there exist an open set containing x not containing y and there exist an open set containing y not containing x.
T_2**(Hausdorff space)**: For distinct x and y in X there exist a disjoint pair of open set U containing x and V containing y.
T_3**(Regular space)** : T_1 Space and for a closed set in A in X and x not in A there exist a disjoint pair of open set U containing A and V containing x.
$T_{3\frac{1}{2}}$**(Completely regular space)**: T_1 Space and for a closed set A and x not in A there exist a continuous function f, $f : X \to [0,1]$ such that $f(x) = 1$ and $f(A) = 0$.
T_4**(Normal space)**: T_1 Space and for two disjoint closed sets A and B there exist a continuous mapping f, $f : X \to [0,1]$ such that $f(A) = 1$ for all a in $A$and $f(B) = 0$ for all b in B.
$T_{4\frac{1}{2}}$**(Completely Normal space)**: X is completly normal if and only if for every pair A, B of separated sets in X (closure of one doesnot intersect another), there exist disjoint open sets containing them **or** equivalently X is said to be completely normal if every subspace of X is normal.
We can observe the following arrow diagram:
T_0 **space** $\to$ T_1 **Space** $\to$ **Hausdorff space** (T_2) $\to$ **Regular space(**T_3**)** $\to$ **Completely regular space** ($T_{3\frac{1}{2}}$) $\to$ **Normal space** (T_4) $\to$ **Completely space(** $T_{4\frac{1}{2}}$**).** In the following segment, we have some basic observations:

- In the above arrow diagram, each member in the right hand side stronger than its left ones.
- Separation by continuous mapping implies separation by open sets.
- All the properties in separation axiom upto completely regular are hereditary and closed under product.
- Every compact Hausdorff spaces are normal.

Lets discuss some important mappings as follows:
Homeomorphism (Topological Equivalence) A mapping f is said to be homeomorphism if

- f is one one and onto.

- f is continuous.
- f^{-1} continuous.

Embedding (Topological Insertion) A mapping f is said to be embedding if

- f is one one.
- f is continuous.
- f is open or closed mapping.

An Embedding can be seen as homeomorphism, setting range as its co-domain.
Automorphism A mapping f is said to be automorphism if it is a topological homeomorphism from a topological space X to itself.

3 Compactification background

Lets define compactification formally:

3.1 Definition

Let X be topological space. A Compactification of X is a compact hausdorff space Y containing X as a subspace such that X is dense in Y. And two Compactifications Y^1 and Y^2 of X is said to be equivalent if there is a homeomorphism $h : Y_1 \to Y_2$ such that $h(x) = x$ for every x in X.
Example : For a, b in $\mathbb{R}$, (a, b) is contain in compact hausdorff space $[a, b]$. And $\overline{(a, b)} = [a, b]$. Hence $[a, b]$ is a compactification of (a, b).
As we can see, being a subset of a compact Hausdorff space (or normal and hence completely regular), any compactifiable space necessarily be a completely regular space.

It is very much worthy to talk about intuitive or basic kind of compactification, which actually enlighten the path of Compactification theory, the Alexandorff one point compactification.

3.2 Motivation and History of One-Point Compactification

In order to move past the naive intuition behind the historical notion of finiteness, the mathematical community required the leadership and direction of mathematicians such as Bernhard Riemann, John Von Neumann and Marshall Stone. Riemann (1826-1866) provided the first example of a compactification using the intuitive infinite and non-compact topological space $\mathbb{C}$ with the construction of the Riemann sphere in 1858-1859. The Riemann sphere is what is known as the one point compactification (defined later) of the complex plane $\mathbb{C}$ - it adds one point to represent infinity at the north pole and transform the plane to the surface of the 2- sphere S^2.

$$\lambda = \frac{(x+iy)}{(1-z)}\,,\ \lambda \in \mathbb{C},$$
$$\text{where (x,y,z)} \in \{(x,y,z) : x^2 + y^2 + z^2 \ \leq 1\} \setminus \{(0,0,1)\}$$

This simple and elegant example of compactification intrigued mathematician and physicists, because it allowed for the reduction of seemingly impossible complex analytic problems on S^2. Subsequently, mathematicians were interested in generalizing the notion of compactification to any topological space. And after that many fascinating ideas arose from different field of studies, which leads to many unnoticed construction in mathematical history. Among them the simplest but fascinating one is Alexandorff one-point compactification.

3.3 Definition

If Y is a compact hausdorff space and X is a proper subspace of Y whose closure equals Y, then Y is said to be a compactification of X. If $Y \setminus X$ equals a single point, then Y is called the one -point compactification of X.

Construction of Alexandroff one point compactification need some basic property of a topological space, known as local compactness.

Local Compactness: A space X is said to be locally compact at x if X has a subspace which is compact neighborhood of x.

Example 1: $\mathbb{R}$ is locally compact. The point lies in some interval (a,b), which in turn is contained in the compact subspace [a,b].

Example 2: $\mathbb{Q}$ is not locally compact space .

3.3.1 Necessary and Sufficient Condition for a Space to Possess "One-Point Compactification"

Theorem 3.1. *Let X be a space. Then X is locally compact Hausdorff if and only if there exists a space Y satisfying the following condition :*
(1) X is subspace of Y.
(2) The set $Y \smallsetminus X$ consists of one point.
(3) Y is compact hausdorff space.

if Y and Y' are two space satisfying these conditions, then there is a homeomorphism of Y and Y' that equals the identity map on X.

Proof :(Munkers) Step 1. We first verify uniqueness. Let Y and Y' be two spaces satisfying these conditions. Define $h: Y \to Y'$, and letting h map the single point of $Y \smallsetminus X$ to the point q of $Y' \smallsetminus X$, and letting h equal the identity on X. We show that if U is open in Y, then $h(U)$ is open in Y'. Symmetry then implies that h is homeomorphism.

First, consider the case where U doesn't contain p. then $h(U) = Y$. Since U is open in Y and is contained in Y, it is open in X. Because X is open in Y', the set U is also open in Y', as desired.

Second, suppose that U contains p since $C = Y \smallsetminus U$ is closed in Y, it is compact as a subspace of Y. Because C is contained in it is an compact subspace of X. Then because X is a subspace of Y', the space C is also a compact subspace of Y'. Because Y' is Hausdorff, C is closed in Y', so that $h(U) = Y' \smallsetminus C$ is open in Y' as desired.

Step 2. Now we suppose X is locally compact Hausdorff and construct the space Y. Step 1 gives us an idea how to proceed. Let us take some object that is not a point of X, denote it by the symbol ∞ for convenience, and adjoint it to X, forming the set $Y = X \cup \{\infty\}$. Topologize Y by defining the collection of open sets of Y to consists of (1) all sets U that are open in X, and (2) all sets of the form $Y \smallsetminus C$, where C is a compact subspace of X.

We need check that this collection is, in fact, a topology on Y. The empty set is a set of type (1), and the space Y is a set of type (2). Checking that the intersection of two open sets is open involves three cases:

$(U_1 \cap U_2)$ is of type (1)

$(Y \smallsetminus C_1) \cap (Y \smallsetminus C_2) = Y \smallsetminus (C_1 \cup C_2)$ is of type (2).

$U_1 \cap (Y \smallsetminus C_1) = U_1 \cap (X \smallsetminus C_1)$ is of type (1) .

because C_1 is closed in X. Similarly, union of any collection of open sets is open :
$\cup U_\alpha = U$ is of type (1)
$\cup(Y \smallsetminus C_\beta) = Y \smallsetminus (\cap C_\beta)$ is of type (2)
$(\cup U_\alpha) \cup (\cup (Y \smallsetminus C_\beta)) = U \cup (Y \smallsetminus C) = Y \smallsetminus (C \smallsetminus U)$,
which of type (2), because $C \smallsetminus U$ is a closed subspace of$C$ and therefore compact.
Now we show X is subspace of Y. Given any open set in Y, we show its intersection with X is open in X. If U is of type (1), then $U \cap X = U$; if $Y \smallsetminus C$ is of type (2), then $(Y \smallsetminus C) \cap X = X \smallsetminus C$; both of these sets are open in X. Conversely, any set open in X is of type (1) and therefore open in Y by definition.
To show: Y is compact let ζ be an open covering in Y. The collection ζ must contain an open set of type (2), say $Y \smallsetminus C$, since none of the open sets of type (1) contain the point $'\infty'$. Take all the members of ζ different from $Y \smallsetminus C$ and intersect them with X; they form a collection of open sets of X covering C. Because C is compact finitely many of them cover C; the corresponding finite elements of ζ will, along with the elements of $Y \smallsetminus C$ are disjoint neighborhood of X and ∞, respectively, in Y.
Step 3. Finally, we prove the converse. Suppose a space Y satisfying conditions (1)-(3) exists. Then X is Hausdorff because it is subspace of a Hausdorff space Y. Given $x \in X$, we show X is locally compact at X. Choose disjoint open sets U and V of Ycontaining x and the single point of $Y \smallsetminus X$, respectively. The the set $C = Y \smallsetminus V$ is closed in Y, so it is compact subspace of Y. Since C lies in X, it is also compact subspace of X; it contains the neighborhood U of$X$.

Here, locally compactness compactness of original space need necessarily and sufficiently for hausdorff property of Y.

It is always convenient to allow topological embedding rather than insist that the original be actually a subspace of the constructed compact hausdorff space. The following lemma will establish the fact "An Proper embedding give rise to a compactification".

Lemma 3.2. *Let X be a space; suppose $h : X \to Z$ is an embedding of X in the compact hausdorff space Z. Then there exists a corresponding compactification Y of X; it has the property that there is an embedding $H : Y \to Z$ that equals h on X. The compactification Y is uniquely determined up to equivalence.*

Keeping this in mind, we restate our definition of compactification as bellow: A compactification of a topological space X is defined as a pair (f, Y), where Y is a compact hausdorff space and f is homeomorphism of X onto a dense subspace of y.

3.4 Some Important Examples

One point compactification of the Real Line The circle S^1 can be viewed as a compactification of the real line. Let h be the inverse projection pictured below: here $h[R] = S^1$-North Pole. We can think $h[R]$ as a 'bent' topological copy of R, and the compactification is created by tying together the two ends of R by adding one new point at infinity (the North Pole).

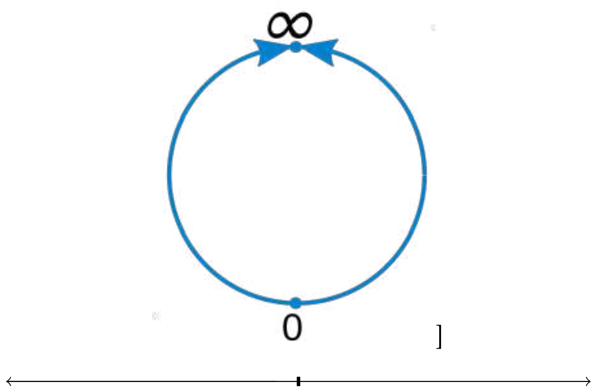
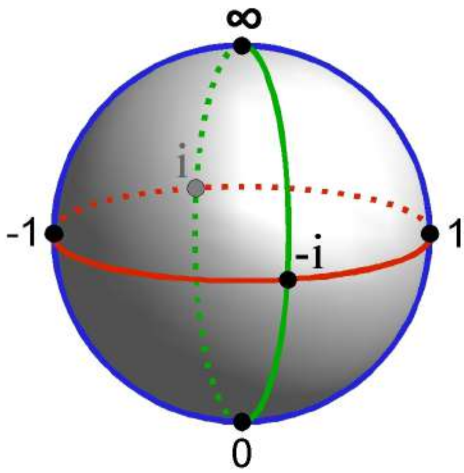


fig : One Point Compactification of Real Line

One point compactification of the Complex Plane As stated the Riemann sphere S^2 is the one point compactification of the plane $\mathbb{C}$.it adds one point to represent infinity at the north pole and transform the plane to the surface of the 2- sphere S^2.

$$\lambda = \frac{(x+iy)}{(1-z)}\ ,\ \lambda \in \mathbb{C},$$
where (x,y,z) $\in \{(x, y, z) : x^2 + y^2 + z^2 \ \leq 1\} \smallsetminus \{(0, 0, 1)\}$



One point compactification of the Natural Numbers The one point compactification of $\mathbb{N}$ is homeomorphic to the subspace $0 \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ of $\mathbb{R}$. The one-point compactification of $\mathbb{N}$ consists of $\mathbb{N}$ together with a single point which we can call ∞. The topological structure is that of the discrete topology on $\mathbb{N}$; and the open neighborhoods of ∞ are by definition the complements of the compact subsets of $\mathbb{N}$. The compact subsets of $\mathbb{N}$ are the finite subsets, so the neighborhoods of ∞ are the sets with finite complement. The map $n \to \frac{1}{n}$ (where $\frac{1}{\infty}$ is interpreted as 0) takes this space to the set $K = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$, and it preserves the topology because the points $\frac{1}{n}$ are all isolated, and the neighborhoods of 0 are exactly the sets with finite complement.

3.5 n-Point Compactification

If Y is a compactification of X and $Y \setminus X$ contains n points then compactification is called n - point compactification.
For example: $\mathbb{R} \cup \{-\infty, \infty\}$ is an 2-point compactification of $\mathbb{R}$.

4 Čech-Stone Compactification

We have seen one point compactification of a locally compact hausdorff space and it seems much intuitive and observable. Now we will see a nice abstract construction of compactification, which is not seem that intuitive.

4.1 Motivation

This idea of compactification arise from physical science background. John Von Neumann (1903-1957) study of ring of bounded continuous function for his axiomatic formulation of quantum mechanics inspired the notion of maximal compactification for a space (Murray 1988). In quantum mechanics, each function in the ring represent potential energy curve of a particle in the phase space X and trivial reality is that no particle has unbounded potential energy. A discrete state that a particle is in, depends intrinsically on the particle's potential energy. Therefore if one discovers that X is viable space for a quantum model then he or she would like to construct maximal phase space Y such that X can be embedded in Y such that all function on Y are bounded and continuous.

Von Neumann intuitively realized this and provided the analytic foundations for this what is now known as C^*- algebra of a topological space X, $C^*(X)$. Later Alexandorff, Urysohn works on systematic study of compactness and Tikhonov's paper on compactification of completely regular spaces in product spaces (1930). Having all these motivations and ideas, In 1937 Eduard Čech and M. H. stone independently found a compactification of a completely regular space and later it was named Čech - Stone compactification(Walker). Here we will follow generalized Čech method of working to construct the Čech - Stone compactification.

4.2 Čech-Stone Compactification

For each Tychonoff space X, we have a compactification $(e, \beta(X))$ where $e : X \to \beta(X)$ is an embedding such that $e(X)$ is dense in $\beta(X)$ and call it the Čech-Stone compactification. It is characterized by the fact that any continuous map $f : X \to C$ of X into a compact hausdorff space C extends uniquely to a continuous map $g : \beta(X) \to C$. As per mentioned in the previous line, our main interest in the Čech-Stone Compactification stems from the extension property it has. Let X be a Tychonoff space, Y is some other space and suppose $f : X \to Y$ is continuous. Let $(e, \beta(X))$ be the Čech - Stone Compactification of X. By means of embedding e, we identify X with $e(X)$, then f can be regarded as a map from $e(X)$ to Y. If we don't identify X with $e(X)$ then the problem amounts to asking whether there exists a map $g : \beta(X) \to Y$ such that $g \ o \ e = f$.

However if the space Y is Hausdorff and compact a solution exists and unique.

4.3 Why Completely Regular ??

If Y is a Čech-Stone compactification of X, then X need to be completely regular. So lets find why is it necessary and sufficient condition for X to be completely regular and Hausdorff. Searching for this condition needs some ideas , lets state them as definitions for X be topological space and Y is the Čech-Stone compactification of X, Y is compact hausdorff.
$C^*(X)$: the ring of all bounded real valued continuous functions on X.
C^*- embedding : A subspace S of X is said to be C^*- embedded in X if every function f in $C^*(S)$ extends to a function g in $C^*(X)$.
Now, X is a topological space C^*-embedded in Y, compact hausdorff. So, question arise what topological constraints should we put in X such that it can be C^*- embedded in Y (necessarily and sufficiently). Urysohn (1898-1924) came with a answer to the question with Urysohn extension theorem.
Urysohn Extension Theorem(UET) :
Definition : Two subsets A and B of a space X are completely separated in X if there exist a continuous mapping f in $C(X)$ such that $f(a) = 0$ for all a in A and $f(b) = 1$ for all b in B.
Theorem (UET) : A subspace S of a space X is C^*- embedded in X if and only if any two completely separated set in S are completely separated in X.
Urysohn extension theorem gives us crucial requirement for our desired compactification. Namely we need Y to have some collection of subsets that are completely separated. As it turns out, we would like the closed subsets of Y to be completely separated. Andrey Tikhonov (1906-1993) made this choice in 1930 since he was looking for a way to preserve compactification for subspace of Y. X is C^*- embedded in Y if and only if two completely separated closed sets in X are completely separated in Y. As closed subspace of compact space are compact, the closed subspace of Y provide a good hint of where to start in the compactification process.
In 1931, Tikhonov proved following theorem :
Tychonoff's theorem: The completely regular space are those spaces which can be embedded in a product of copies of the closed unit intervals $I = [0,1]$.

To see why this theorem is extremely important let us recall Urysohn Lemma, this lemma shows an equivalence.

Urysohn's Lemma : X is normal if and only if A, B disjoint closed sets in X the there exists a continuous function $f : X \to [0,1]$ such that $f(a) = 0$ for all a ∈ A and $f(b) = 1$ for all b ∈ B.

As compact hausdorff spaces are normal(Munkers, theorem 32.3) and one points sets are closed. So above 2 statements give rise to very important topological information of space X.

No larger class of topological spaces can be studied by means of C^* ***-embedding into compact hausdorff space.***

Thats why our compactifiable space necessary and sufficiently completely regular and hausdorff or Tychonoff space. (completely regular+ hausdorff space was named Tychonoff space in the name of Russian Mathematician Andrey Nikolayevich Tikhonov, a student of Pavel Alexandorv, because of his huge contributions to these theories.)

4.4 Construction of Čech-Stone Compactification

There are some technical requirements that are necessary for a map from a space X to spaces indexed by family F of functions to be an embedding; These technicalities are precisely what the embedding lemma discern. We will deal with function that preserves the completely separable structure of completely regular spaces, which are described by the following families of functions.

Definition : A family F of functions on X is said to distinguish points if for each pair of distinct points x and y, there exists an f in F such that $f(x) \neq f(y)$. A family F of function is said to be distinguish points and closed sets if each closed set A in X and each point x not in A, there exist a mapping f in A such that $f(x) \notin \overline{(f(A))}$.

Lemma : (Embedding Lemma) Let F be a family of mapping such that each member f in F maps $X \to Y_f$. Then mapping $e : X \to \prod Y_f$ defined by $(\pi_f \ o \ e)(x) = f(x)$ for all x in X is continuous. Then

a) The evaluation mapping $e : X \to \prod Y_f$ defined by $(\pi_f o e)(x) = f(x)$ for all x in X is continuous.

b) The mapping e is an open mapping onto $e(x)$ if F distinguishes points and closed sets.

c) The mapping e is one-one iff F distinguishes points.

d)The mapping e is an embedding if F distinguishes points and distinguishes points and closed sets.

Proof :

a) Let $\pi_g : \Pi_{f\in F} Y_f \to Y_g$ be the projection map to the space Y_g. Then $\pi_g\ o\ e = g$ so that $\pi_g o\ e$ is continuous. Therefore e must be continuous as g is continuous.

b) Suppose that U is open in X and $x \in U$. Choose $f \in F$ such that $f(x) \notin \overline{f(X \smallsetminus U)}$. The set $B = \{z \in e(X)|\pi_f(z) \notin f(X \smallsetminus U)\}$ is a neighborhood of $e(x)$ as the set is open (it is defined for components not being in the closed set $f(X \smallsetminus U)$) and clearly $e(x) \in B$. Moreover, $\pi_f(B) \subset f(U)$ by construction. It is now claimed that $f(U) \subset \pi_f(B)$. This follows trivially from the definition of a family of functions distinguishing points and closed sets therefore $f(U) = \pi_f(B)$ and subsequently $f(U)$ is a open subset of $(\pi_f\ o\ e)(X)$. Therefore the evaluation mapping is an open mapping.

c) The definition of distinguishing points implies injectivity of the evaluation function e.

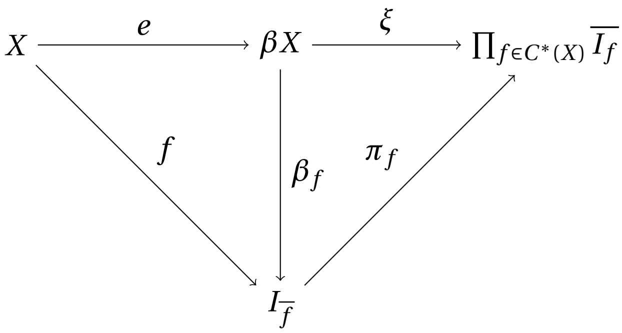
d) Combining (a), (b) and (c) we see that $X \cong e(X)$ as e is continuous, open, injective, surjective map.

4.5 Existence and Uniqueness

We have developed the tools to finally describe the Čech-Stone compactification for a completely regular space and it is stronger result then Tychonoff's theorem for completely regular spaces.

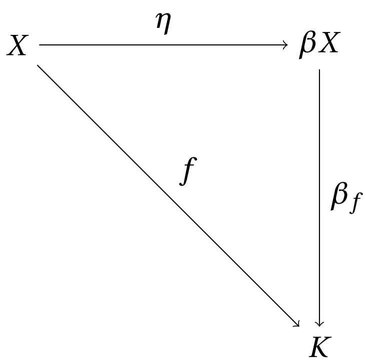
Existence : Every completely regular space X has a hausdorff compactification βX in which it is C^*- embedded.

Proof : For each $f \in C^*(X)$, Let I_f denote the range of f. As f is bounded $\overline{I_f}$ must be compact by Heine-Borel. Now we wish to apply the embedding lemma: let $F = C^*(X)$; Then X can be embedded in $\prod_{f\in C_*(X)} \overline{I_f}$ by the evalution map $\pi_f o e(x) = f(x)$. Now let $\beta X = \overline{e(X)}$; as $e(X) \subset \prod_{f\in C_*(X)} \overline{I_f}$, there exists a trivial and closed embedding of $\beta X \in \prod_{f\in C_*(X)} \overline{I_f}$, say ξ. Finally, define the map $\beta_f : \beta X \to \overline{I_f}$ by $\beta_f = \pi_f o\xi$. This gives us the following commutative diagram:



From this construction, we see that if X is compact, then $\beta X \cong X$. The maximality of the construction is implicit and comes from the fact that we are embedding X into compact space indexed by all continuous, bounded function on X. Note that the uniqueness of the Čech-Stone compactification was proved by Marshell Stone whereas the existence of the extension was proved by Stone and Čech independently.

Uniqueness Part : (Every completely regular space X has a compacatification βX such that any mapping of X to a compact space K will extend uniquely to βX). In terms of commutative diagram we have :



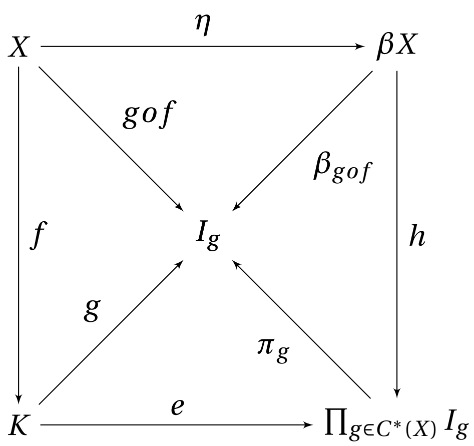
Proof: As in the previous theorem, for each $g \in C^*(K)$, let I_g represent the range of K and let $e : K \to \prod_{g\in C^*(K)} I_g$ be the evaluation map. As $g\ o\ f(X) \subset I_g$, the Čech-Stone existence theorem provides that there exists a extension β_{gof} of $(g\ o\ f)$ to βX.

Claim 1: f uniquely extends to βX.

Suppose we define a mapping $h : \beta X \to \prod_{g\in C^*(K)} I_g$ such that

$\pi_g oh(p) = \beta_{gof}(p)$

for all $p \in \beta X$ .Then h is continuous as β_{gof} is continuous (by construction) and subsequently this forces the above composition to be continuous in each I_g. This gives us the following commutative diagram :



As e is an embedding, by construction, we need to show that $h(\beta X) \subset e(K)$ in order to show that h embeds η in $\prod_{g\in C^*(X)} I_g$. Note that for each $g \in C^*(K), \beta_{gof}(X) \subset g(K)$ so that we have :
$h(\beta X) = h(cl_{\beta X}(X)) \subseteq \overline{e(K)} = cl(K) = K$
where the last equality holds from the compactness of K. Finally, the uniqueness of the extension h comes directly from the fact that any two extension of f are equivalent on βX.

4.6 Important Compactifications

To establish examples of compactification, we need the concept of filter and ultra filter.
Filters and Ultrafilters A filter on a set X is a non empty family F of subsets of X such that
1. $\emptyset \notin F$.
2. F is closed under finite intersections.
3. If $B \in F$ and $B \subset A$ then $A \in F$ for all $A, B \subset X$.
A filter is an ultrafilter if it is the maximal element in the collection of all filters on X, partially ordered by inclusion, that is an ultrafilter is not properly contained in any filter.
Construction of $\beta(X)$ when X is discrete : Here we assume X is a discrete topological space, this being the hardest case. Here we simply define $\beta(X) = \{U : U$ is an ultrafilter on $X\}$. Thus points of $\beta(X)$ are just ultrafilters on X. We specify the topology of $\beta(X)$. A

basis for this topology is obtained as follows: for every subset $A \subset X$, define $(A) = \{U \in \beta(X) : A \in U\}$. The subsets (A) form the basis for the topology on $\beta(X)$.

4.6.1 Stone-Čech Compactification of $\mathbb{N}$

For a simple set like $\mathbb{N}$, The Čech-Stone compactification becomes a very complicated object. The present day description of $\beta(\mathbb{N})$ is as the Stone Space of the Boolean Algebra $P(\mathbb{N})$. The underlying set of $\beta(\mathbb{N})$ is the set of all the ultrafilters of $\mathbb{N}$ with the family $\{\overline{X} : X \subset \mathbb{N}\}$ as a base for the open sets, where $\overline{X}$ denotes the set of all the ultrafilters of which X is an element. The space is $\beta(\mathbb{N})$ is separable and its cardinality is the maximum possible that is 2^c. The remainder $\beta(\mathbb{N}) \setminus \mathbb{N}$ denoted by $\mathbb{N}^*$ is an interest of researches.

4.6.2 Stone -Čech Compactification of $\mathbb{R}$

Instead of $\beta(\mathbb{R})$, one generally uses $\beta(H)$; where H is the positive half line $[0,\infty)$. This is because $x \to -x$ induces an automorphism of $\beta(\mathbb{R})$ that shows that $\beta([0, \infty])$ and $\beta([-\infty, 0])$ are the same thing. In a sense $\beta(\mathbb{R})$ looks like $\beta(\mathbb{N})$ in that it is a thin locally compact space with a large compact lump at the end; this remainder H^* has some properties common with $\mathbb{N}^*$. But of course, there are differences like $\beta(\mathbb{R})$ and H^* are connected but $\beta(\mathbb{N})$ and $\mathbb{N}^*$ are most certainly not.

5 Comparision of Compactifications

Let (e, Y) and (f, Z) be two compactifictions of a space X. Then (e, Y) is said to be greater than (f, Z), written as $(e, Y) \geq f, Z)$ if there exists a map $g : Y \to Z$ such that $g \; o \; e = f$.

5.1 The Minimal Compactification

Let (Y_1, h_1) be a one-point compactification of X. For every compactification (Y, h), $(Y, h) \geq (Y_1, h_1)$ We may assume $X \subset Y_1$, $X \subset Y$ and h_1, h are the identity maps. Since X has a one-point compactification, X is locally compact, So X is open in both Y and Y_1 .
Let $Y_1 \setminus X = \{p\}$ and define $f : Y \to Y_1$ by $f(y) = y$ if $y \in X$ and $f(y) = p$ if $y \in Y \setminus X$. To show that $(Y, h) \geq (Y_1, h_1)$, we only need to check that f is continuous in each point

$y \in Y$. If $y \in X$ and V is an open set containing $f(y) = y$ in Y_1, then $y \in U = V \setminus \{p\}$ which is open in X and therefore also open in Y. Clearly, $f(U) = U \subset V$ If $z \in Y \setminus X$ and V is an open neighborhood of $f(z) = p$ in Y_1, then $Y_1 \setminus V = K$ is a compact subset of X. Therefore K is closed in Y, so $U = Y \setminus K$ is an open set in Y containing z and $f(U) \subset V$.

5.2 The Maximal Compactification

Let $(e,\ Y)$ and $(f,\ Z)$ be two compactification of a space X. Then they are said to be topologically equivalent if there exists a homeomorphism $g : Y \to Z$ such that $g\ o\ e = f$.

Proposition: The relation $\geq$ is anti-symmetric upto topological equivalence as far the Hausdorff compactification is concerned.

Among all the Hausdorff compactifications of a Tychonoff space, surprisingly we have the Čech-Stone compactification is largest upto a topological equivalence, Let X be a Tychonoff space and $(e,\ \beta(X))$ be the Čech-Stone Compactification. Now let $(f,\ Y)$ be any compactification of X. Then Y is compact and hausdorff. So by the theorem due to Stone and Čech in the section the map $f : X \to Y$ extends to $\beta(X)$, that is, there exists $g : \beta(X) \to Y$ such that $g\ o\ e = f$. By definition this means that $(e,\ \beta(X)) \geq (f,\ Y)$. Thus the Čech-Stone compactification is the largest. And by the proposition above, upto topological equivalence, the compactification is unique.

6 Conclusion

The set of all compactification of a space X forms a partially ordered set $P_c(X)$ by $\geq$. We have had minimal and maximal compactification in the context of hausdorff compactification, where hausdorff compactifications forms a total ordered subset of $P_c(X)$. Many other compactifications have been constructed for various purposes. Many of them are applicated in different branch of Mathematics such as Probability Theory and in Physical Sciences also. Current days it is regarded as a very active field of research.

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MHD free convection problems

Parisima Phukan
Department of Mathematics, Cotton University, Guwahati, Assam, India

*email:*parismita12007@gmail.com

Abstract In this article,we will discuss the study of some MHD free convection problems. Here we will discuss the unsteady free convective flow of fluid in the presence of radiation ,chemical reactions and variable viscosity. We will mainly focus on the influences of the various parameter namely, Grashof number,Prandtl number, Schmidt number, heat absorption parameter, radiation parameter ,variable viscosity on the velocity,temperature and concentration.

1 Introduction

Fluid dynamics is that branch of science which deals with the study of the motions of fluids or that of bodies in contact with fluids.It is futher classified into several subdisipline including **aerodynamics** and **hydrodynamics**.Aerodynamics is the study of gases in motion and hydrodynamics is the study of liquids in motion.
Magnetohydrodynamics(MHD) is the study of the magnetic properties and behaviour of electrically conducting fluids. The field of MHD was initiated by Hannes Alfven for which he received the Nobel Prize in Physics in 1970. The set of equations which describes MHD are a combination of the **Navier-Stokes equations** of fluid dynamics and **Maxwell's equations** of electromagnetism. The central point of MHD theory is that conductive fluids can support magnetic fields. The presence of magnetic fields leads to the forces that in turn act on the fluid, thereby potentially altering the geometry and

strength of the magnetic fields themselves. A key issue for a particular conducting fluid is the relative strength of the advecting motions in the fluid, compared to the diffusive effects caused by the electrical resistivity.
There are wide applications of MHD in many branches of science and technology such as Astrophysics, Engineering, Geophysics, Medical sciences etc.

2 Continuity equations and Navier-Stokes Equations

In fluid dynamics, the **continuity equations** states that the rate at which mass enters a system is equal to the rate at which mass leaves the system plus the accumulation of mass within the system. The differential form of the continuity equations is

$$\frac{\delta \rho}{\delta t} + \nabla.(\rho u) = 0$$

where ρ is fluid density, t is time, u is the flow velocity vector field.
This equation is also one of the Euler equations.

If the fluid is incompressible i.e ρ is constant, then the equation of continuity takes the form:

$$\nabla.u = 0$$

which means the divergence of the velocity field is zero everywhere.

The **Navier-Stokes** equations are nonlinear partial differential equations which mathematically express the conservation of momentum and conservation of mass.They arise from applying Newton's second law to fluid motion, together with the assumption that the stress in the fluid is the sum of a diffusing viscous term(proportional to the gradient of velocity) and pressure term. The Navier-Stokes equations are expressible in vector form as below:

$$\frac{\delta \vec{u}}{\delta t} + (\vec{u}.\vec{\nabla})\vec{u} = \vec{F} - \tfrac{1}{\rho}\vec{\nabla} p + \nu \vec{\nabla}^2 \vec{u} + \tfrac{\nu}{3}\vec{\nabla}(\vec{\nabla}.\vec{u})$$

where $\vec{F}$ is the body force , $\vec{u}$ is the fluid velocity, ρ is the density, p is the fluid pressure, ν is the kinematic viscosity of the fluid.

3 Prandtl's boundary layer theory

A solution of Navier-Stokes equations is called a velocity field which describes the fluid velocity at a given point in space and time.But these equarions are non-linear partial differential equations ,the exact solution for a particular flow problem of these equations are either difficult or impossible to obtain.
At the beginning of 20th century, the German physicist L.Prandtl introduced the concept of boundary layer theory of fluid flow for small viscosity.He simplified the Navier-Stokes equations to mathematically tractable form which are called the *Prandtl boundary layer equations.*

Prandtl proved theoretically and experimentally that a flow about a solid body can be divided into two regions:

⇒ a very thin layer in the neighbourhood of the body called boundary layer, where friction plays an important role.

⇒ remaining region outside the layer where friction may be neglected.

In boundary layer the velocity of the fluid increases from zero at the wall(no slip) to its full value which corresponds to external frictionless flow.

4 Dimensionless Numbers

The inertia force is the product of mass and acceleration.We know that inertia force always exists in all flow problems. Besides the inertia force, there always exist some additional forces which are responsible for fluid motion.The required conditions for dynamic similarity can always be obtained by considering the ratio of the inertia force and any one of the remaining forces (e.g. viscous force, gravity force, pressure force and so on).Some of the dimensionless numbers are discussed below:

4.1 Reynolds number

The Reynolds number R_e is defined as the ratio of inertia force to viscous force of the flowing fluid i.e

$$R_e = \frac{Inertiaforce}{Viscousforce}$$

If for any flow problem R_e is small then we can ignore the inertia force, whereas if R_e is quite large then we can neglect the effect of viscous force and consequently the fluid may be treated as non-viscous fluid.

4.2 Froude number

The Froude number Fr is defined as the ratio of inertia force to gravity force i.e.

$$Fr = \frac{Inertia\ \ force}{Gravity\ \ force}$$

When the gravity force is predominating, Froude number must be the same for dynamic similiarity of two forces.

4.3 Grashof number for heat transfer

The Grashof number Gr for heat transfer is the ratio of the product of inertia force and buoyancy force to the square of viscous force i.e.

$$Gr = \frac{Inertia\ \ force*Buoyancy\ \ force}{(viscous\ \ force)^2}$$

Grashof number is important in free convection heat transfer where the only driving force is the buoyancy force.In free convection, the flow field is induced by buoyancy forces, which arise from density differences caused by temperature variations in the field.

4.4 Grashof number for mass transfer

The Grashof number Gm for mass transfer is defined as ratio of Buoyancy force due to concentration gradient to the viscous force i.e.

$$Gm = \frac{Buoyancy\ force\ due\ to\ concentration\ gradient}{viscous\ force}$$

Gm is important in free convection involving mass transfer. This is because the density difference in free convection mass transfer is due to species concentration difference and hence the buoyancy force in natural convection mass transfer is incorporated into Gm.

4.5 Prandtl number

The Prandtl number Pr is defined as the ratio of molecular diffusivity of momentum to the molecular diffusivity of heat i.e.

$$Pr = \frac{Molecular\ \ diffusivity\ \ of\ \ momentum}{Molecular\ \ diffusivity\ \ of\ \ heat}$$

The Prandtl number is a measure of value of momentum and heat in the velocity and thermal boundary layer. In other words it measures relative thickness of velocity and thermal boundary layers.

4.6 Schmidt number

The Schmidt number Sc is the ratio of momentum diffusivity to mass diffusivity of the species i.e.

$$Sc = \frac{Momentum \ \ diffusivity}{Mass \ \ diffusivity}$$

It physically relates the relative thickness of velocity and concentration boundary layer.It provides a measure of the relative effectiveness of momentum and mass transport by diffusion in a fluid medium, for a given fluid flow involving convection mass transfer.

5 Study of some MHD free convective problems

The field of MHD was initiated by Hannes Alfven for which he received the Nobel Prize in Physics in 1970.Its wide applications in many branches of science and technology has attracted numerous scientist and engineers for the last several decades.Several authors has made there notable contribution in MHD.
In this subsection we will discuss some MHD free convective problems:

- In 2005, O.D Makinde [7] studied free convection flow with thermal radiation and mass transfer past a moving vertical porous plate.The plate is maintained at a uniform temperature with uniform species concentration and the fluid is considered to be gray, absorbing-emitting.For different parameters involving the problem, the velocity, temperature and concentration distribution is discussed below:

  ⇒ A decrease in suction parameter, Schmidt Number and Prandtl Number will enhance the fluid velocity.

⇒ An increase in Grashof number and thermal radiation intensity will enhance the fluid velocity and the boundary layer.

⇒ An increase in thermal radiation interaction and a decrease in the fluid suction at the plate will enchance the fluid temperature.

⇒ A decrease in fluid suction and Schmidt Number will enhance the fluid concentration.

- D.Sharma,N.Ahmed,H.Deka [8] have studied the MHD free convection and mass transfer flow past an accelerated vertical plate with chemical reaction in presence of radiation.By assigning some selected values of parameters , velocity, temperature and rate of heat and mass transfer are investigated.Below are some of the results obtained:

⇒ Velocity decreases with the increase in magnetic parameter.

⇒ The radiation parameter decreases the temperature distribution in the thermal boundary layer.

⇒ The increase in radiation parameter leads to decrease the boundary layer thickness and to enhance the heat transfer rate in the presence of thermal buoyancy force.

⇒ Due to the effect of radiation the rate of heat transfer from the plate to the fluid gets increased.

⇒ Schmidt number and Chemical reaction parameter increases the rate at which the mass gets transfered.

- P.Ramakrishna Reddy and M.C. Raju [5] have studied unsteady free convective flow of a double diffusive fluid past a moving vertical porous plate in the presence of thermal radiation and first order homogeneous chemical reaction.Using perturbation technique (Chamkha [2], P.Ram,A.Kumar and H.Singh [4]),the solution of PDE along with boundary condition is obtained to investigate the effects of various parameters on velocity and temperature fields in the boundary layer generated on the surface.Below are some of the results obtained:

  ⇒ As Prandtl number increases ,temperature of fluid decreases.

  ⇒ The temperature profiles have significant appearance near the plate for all values of Reynolds number but as the values of Reynolds number increases, temperature decreases and reaches ambient temperature far away from the plate.

  ⇒ Temperature of the fluid decreases as the radiation parameter increases.

  ⇒ Temperature increases for increasing values of absorption parameter but reverse trend in the presence of generation parameter.

  ⇒ Increasing value of Schmidt number decreases concentration

  ⇒ Increasing value of chemical reaction parameter decreases the concentration.

  ⇒ Velocity increases on increasing the values of thermal Grashof number.

  ⇒ Velocity increases on increasing the values of Reynolds number(Re).

- R .Biswas and M,Afikuzaman [6] have studied the MHD free convection and heat transfer fluid flow through a semi-infinite vertical porous plate with the effects of chemical reaction.An explicit finite difference technique has been used to obtain the numerical results of the problem and the physical situation has been carried out for the different values of various parameters on velocity, temperature and concentration through a vertical porous permeable plate within the boundary.Below are some of the results obtained:

⇒ Velocity decreses with the increase of magnetic parameter.

⇒ Velocity decreases due to the increase of chemical reaction.

⇒ Velocity increases due to the increase of Grashof number(Gr).

⇒ Velocity decreases with the increase of magnetic parameter and Prandtl number.

⇒ Temperature decreases with the increase in Schmidt number

⇒ Temperature increases with the increase of heat source parameter

⇒ Chemical reaction decreases concentration profiles.

6 Conclusion

As we have seen for an unsteady free convective flow of fluid, temperature,velocity and concentration has been influence by various parameters.Those results that are obtained will help researchers to do further research in MHD convection problems.

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A survey on construction of irreducible polynomials over finite fields

Apolo Chutia
Department of Mathematical Sciences, Tezpur University, Sonitpur, Assam, India

email: apolochutia111@gmail.com

Abstract. In this article we will briefly discuss two methods of construction of irreducible polynomials over finite fields, i.e. construction by composition and construction by itteration. Also we will discuss some results related to them.

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Keywords. Irreducible polynomial, Composite polynomial, Finite field.

1 Construction by composition

Known constructions of irreducible polynomials depend on the composition of an initial irreducible polynomial with a further polynomial or rational function. Often this process can be iterated or continued recursively to produce an infinite sequence of irreducible polynomials of increasing degrees. The following theorem was employed by several authors to give iterative constructions of irreducible polynomials over finite fields.

Theorem 1.1. *[4, 5] Let, $f, g \in \mathbb{F}_q[y]$ be coprime polynomials and let $Q \in \mathbb{F}_q[y]$ be an irreducible polynomial of degree n. Then the composition*

$$H(y) = g(y)^n Q\Big(\frac{f(y)}{g(y)}\Big)$$

is irreducible over $\mathbb{F}_q$ if and only if $f - \alpha g$ is irreducible for any zero $\alpha \in \mathbb{F}_{q^n}$ of Q.

A further extension of the theorem is produced in [14], which is also instrumental in the construction of irreducible polynomials of relatively higher degree from given ones.

Theorem 1.2. *[10] Let $Q \in \mathbb{F}_q[y]$ be irreducible of degree n. Then for any $a, b, c, d \in \mathbb{F}_q$ such that $ad - bc \neq 0$,*

$$H(y) = (cy + d)^n Q\Big(\frac{ay + b}{cy + d}\Big)$$

is also irreducible over $\mathbb{F}_q$.

Theorem 1.3. *[10] Let t be a positive integer and $Q \in \mathbb{F}_q[y]$ be irreducible of degree n and exponent e (equal to the order of any root of Q). Then $P(y^t)$ is irreducible over $\mathbb{F}_q$ if and only if*

(a) $(t, (q^n - 1)/e) = 1$

(b) *each prime factor of t divides e, and*

(c) *if* $4|t$ *then* $4|(q^n - 1)$.

Agou [3] has established a criterion for $f(g(y))$ to be irreducible over $\mathbb{F}_q$, where $f, g \in \mathbb{F}_q[y]$ are monic and f is irreducible over $\mathbb{F}_q$. This criterion was used in Agou [3] to characterize irreducible polynomials of special types such as $f(x^{p^r} - ax)$, $f(x^p - x - b)$ and others. Such irreducible compositions of polynomials are also studied in Cohen [5], Long, and Ore . Irreducibility criteria for compositions of polynomials of the form $f(x^t)$ have been established by Agou [1, 2, 3], Butler, Cohen, Pellet, Petterson, and Serret. Berlekamp and Varshamov and Ananiashvilii discussed the relationship between the orders of $f(x^t)$ and that of $f(y)$.

Theorem 1.4. *[11] Let q be an odd prime power. If P is an irreducible polynomial of degree n over $\mathbb{F}_q$, then $x^n Q(x+x^{-1})$ is irreducible over $\mathbb{F}_q$ if and only if the element $P(2)P(-2)$ is a non-square in $\mathbb{F}_q$.*

We briefly describe some constructive aspects of irreducibility of certain types of polynomials, particularly binomials and trinomials.

Definition 1.5. *[14] A binomial is a polynomial with two nonzero terms, one of them being the constant term.*

Definition 1.6. *[14] A trinomial is a polynomial with three nonzero terms, one of them being the constant term.*

Irreducible binomials can be characterized explicitly. For this purpose it suffices to consider nonlinear, monic binomials.

Theorem 1.7. *Let $a \in \mathbb{F}_q$ and let p be the characteristic of $\mathbb{F}_q$. Then the trinomial $x^p - x - a$ is irreducible in $\mathbb{F}_q[y]$ if and only if it has no root in $\mathbb{F}_q$.*

The fact that $x^p - x - a$ is irreducible over $\mathbb{F}_p$ if $a \in \mathbb{F}^*{}_p$ was already established by Serret [2597, 2600]. See also Dickson [6], [[6], Part I, Chapter 3] and Albert [, Chapter 5] for these results.

If we consider more general trinomials of the above type for which the degree is a higher power of the characteristic, then these criteria need not be valid any longer. In fact, the following decomposition formula can be established.

Theorem 1.8. *[14] For $x^p - x - a$ with a being an element of the subfield $K = \mathbb{F}_r$ of $F = \mathbb{F}_q$ we have the decomposition*

$$x^q - x - a = \prod_{j=1}^{q/r} (x^r - x - \beta_j)$$

in $\mathbb{F}_q[y]$, *where the* β_j *are the distinct elements of* $\mathbb{F}_q$ *with* $Tr_{F/K}(\beta_j) = a$.

Theorem 1.9. *[12, 13] Let* $P(y) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ *be an irreducible polynomial over the finite field* F_q *of characteristic p and let* $b \in \mathbb{F}_q$ *. Then the polynomial* $P(x^p - x - b)$ *is irreducible over* $\mathbb{F}_q$ *if and only if the absolute trace* $Tr_{F/K}(nb - a_{n-1} \neq 0$.

The above theorem was shown in this general form by Varshamov [12, 13]; see also Agou [3]. The case $b = 0$ received considerable attention much earlier. The corresponding result for $b = 0$ and finite prime fields was stated by Pellet and proved in Pellet [14]. Polynomials $f(x^p - x)$ over $\mathbb{F}_p$ with deg(f) a power of p were treated by Serret. The case $b = 0$ for arbitrary finite fields was considered in Dickson [850], Part I, Chapter 3] and Albert [70, Chapter 5]. More general types of polynomials such as $f(x^p - ax), f(x^p - ax - bx)$ and others have also been studied, see Agou [1, 2, 3], Cohen [5], Long, Long and Vaughan, and Ore.

2 Recursive constructions

Theorem 2.1. *[8] Let* $q = p^s$ *be a prime power and let* $f(y) = \sum_{u=0}^{n} c_u x^u$ *be a monic irreducible polynomial over* $\mathbb{F}_q$ *. Denote* $\mathbb{F}_q = F$ *and* $\mathbb{F}_p = K$*. Suppose that there exists an element* $\delta_0 \in \mathbb{F}_q$ *such that* $f(\delta_0) = a$ *with* $a \in \mathbb{F}^*{}_p$ *, and*

$$Tr_{F/K}(n\delta_0 + c_{n-1}.Tr_{F/K}(f'(\delta_0))) \neq 0,$$

where f' is the formal derivative of f.

Theorem 2.2. *[8, 9] Let* $\delta \in \mathbb{F}^*{}_{2^n}$ *and* $f_1(y) = \sum_{u=0}^{n} c_u x^u$ *be a monic irreducible polynomial over* $\mathbb{F}_{2^n}$*, whose coefficients satisfy the conditions*

$$Tr_{F/K}\Big(\frac{c_1\delta}{c_0}\Big) = 1$$

and

$$Tr_{F/K}\Big(\frac{c_{n-1}}{\delta}\Big) = 1,$$

where $\mathbb{F}_{2^n} = F$ *and* $\mathbb{F}_2 = K$. *Then all the terms in the sequence* $(f_k(y))_{k\geq 1}$ *defined as*

$$f_{k+1}(y) = x^{2^{k-1}n} f_k(x + \delta^2 x^{-1}), k \geq 1,$$

are irreducible polynomials over $\mathbb{F}_{2^n}$.

Theorem 2.3. *[8] Let* $f(y) = \sum_{i=0}^{n} cix^i$ *be irreducible over* $\mathbb{F}_{2^m}$ *of degree* n. *Denote* $\mathbb{F}_{2^m} = F$ *and* $\mathbb{F}_2 = K$. *Suppose that* $Tr_{F/K}(c_1/c_0) \neq 0$ *and* $Tr_{F/K}(c_{n-1}/c_n) \neq 0$. *Define the polynomials* $a_k(y)$ *and* $b_k(y)$ *recursively by* $a_0(y) = x, b_0(y) = 1$ *and for* $k \geq 1$

$$a_{k+1}(y) = a_k(y) b_k(y),$$

$$b_{k+1}(y) = a^2{}_k(y) + b^2{}_K(y).$$

Then

$$f_k(y) = (b_k(y))^n f(a_k(y)/b_k(y))$$

is irreducible over $\mathbb{F}_{2^m}$ *of degree* $n2^k$ *for all* $k \geq 0$.

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Topological entropy for non-compact sets and its recent extensions

Rupam Gogoi

Department of Mathematical Sciences, Tezpur University, Sonitpur, Assam, India

email: rupamgogoi97@gmail.com

Abstract. In the year 1965, R. L. Adler, A. G. Konheim and M. H. McAndrew introduced the notion of topological entropy for compact topological spaces. They introduced entropy as an invariant for continuous mappings. In 1973 Rufus Bowen introduced the topological entropy for non-compact sets. Here we present the topological entropy for non-compact sets and some recent extensions and work on this.

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Keywords. Topological Entropy, Hausdorff dimension, Homeomorphisms.

1 Introduction

Let $f : X \to X$ be continuous and $Y \subset X$. The topological entropy $h(f,Y)$ will be defined much like Hausdorff dimension, with "size" of a set reflecting how f acts on it rather than its diameter. Let $\mathscr{G}$ be a finite open cover of X. We write $E \prec \mathscr{G}$ if E is contained in some member of $\mathscr{G}$ and $\{E_i\} \prec \mathscr{G}$ if every $E_i \prec \mathscr{G}$. Let $n_{f,\mathscr{G}}(E)$ be the biggest non-negative integer such that

$$f^k E \prec \mathscr{G} \, for \, all \, k \in [0, n_{f,\mathscr{G}}(E));$$

$n_{f,\mathscr{G}}(E) = 0$ if $E \not\prec \mathscr{G}$ and $n_{f,\mathscr{G}}(E) = +\infty$ if all $f^k E \prec \mathscr{G}$. Now set

$$D_G(E) = exp(-n_{f,\mathscr{G}}(E))\, and\, D_{\mathscr{G}}(\xi, \lambda) = \Sigma_{i=1}^{\infty} D_{\mathscr{G}}(E_i)^{\lambda}$$

for $\xi = \{E_i\}_{i=1}^{\infty}$ and $\lambda \in R$. Define a measure $m_{\mathscr{G},\lambda}$ by

$$m_{\mathscr{G},\lambda}(Y) = lim_{\epsilon \to 0} inf\{D_{\mathscr{G}}(\xi, \lambda) : \cup E_i \supset Y\, and\, D_{\mathscr{G}}(E_i) < \epsilon\}.$$

Notice that $m_{\mathscr{G},\lambda}(Y) \leq m_{\mathscr{G},\lambda'}(Y)$ for $\lambda > \lambda'$ and $m_{\mathscr{G},\lambda}(Y) \notin \{0, +\infty$ for at most one λ. Define

$$h_{\mathscr{G}}(f, Y) = inf\{\lambda : m_{\mathscr{G},\lambda}(Y) = 0\}\, and\, finally h(f, Y) = sup_{\mathscr{G}} h_{\mathscr{G}}(f, Y)$$

where $\mathscr{G}$ ranges over all finite open covers of X. For $Y = X$ we write $h(f) = h(f, X)$.

Proposition 1.1. *If X is compact, then $h(f)$ equals the usual topological entropy.*

Proposition 1.2. *(a) If $f_1 : X_1 \to X_1$ and $f_2 : X_2 \to X_2$ are topologically conjugate for $Y_1 \subset X_1$. (b) $h(f, f(Y)) = h(f, Y)$. (C) $h(f, \cup_{i=1}^{\infty} Y_i) = sup_i h(f, Y_i)$. (d) $h(f^m, Y) = mh(f, Y)$ for $m > 0$.*

Theorem 1.3. *Let $f : X \to X$ be a continuous map of a compact metric space and $\mu \in M(f)$. If $Y \subset X$ and $\mu(Y) = 1$, then $h_{\mu}(f) \leq h(f, Y)$.*

Lemma 1.4. *Let α be a finite Borel partition of X such that every $x \in X$ in the closures of atmost M sets of α. Then*

$$h_{\mu}(f, \alpha) \leq h(f, Y) + log M.$$

Lemma 1.5. *Let $\mathscr{G}$ be a finite open cover of X. For each $n > 0$ there is a finite Borel partition α_n of X such that $f^k \alpha_n \prec \mathscr{G}$ for all $k \in [0, n)$ and at most n card $\mathscr{G}$ sets in α_n can have a point in all their closures.*

Lemma 1.6. *Given a finite Borel partition β and $\epsilon > 0$ there is an open cover $\mathscr{G}$ so that $H_{\mu}(\beta|\alpha) < \epsilon$ whenever α is a finite Borel partition with $\alpha \prec \mathscr{G}$.*

Lemma 1.7. *Let*

$$R(N, m, t) = \{a \in \{1, ..., N\}^m : H(dist\, a) \leq t\}.$$

Then fixing N and t,

$$lim_{m\to\infty} sup \frac{1}{m} log\, card\, R(N,m,t) \leq t.$$

Lemma 1.8. *Suppose $f : X \to X$ is a continuous map of a topological space, $\mathscr{B}$ an open cover of X, β a finite cover of X and M a positive integer so that $f^k\beta \prec \mathscr{B}$ for all $k \in [0,M)$. For $t \geq 0$ define*

$$Q(t,\beta) = \{x \in X : lim_{n\to\infty} inf(inf\{H(q) : q \in Dist_\beta(x,n)\}) \leq t\}.$$

Then $h_{\mathscr{B}}(f, Q(t,\beta)) \leq t/M$.

Theorem 1.9. *Let $f : X \to X$ be a continuous map on a compact metric space. Set*

$$QR(t) = \{x \in X : \exists \mu \in V_f(x)\; with\; h_\mu(f) \leq t\}.$$

Then $h(f, QR(t)) \leq t$.

Corollary 1.10. *Let $f : X \to X$ be a continuous map of a compact metric space. Then*

$$h(f) = sup_{\mu \in M(f)}\, h_\mu(f).$$

Theorem 1.11. *Let f be continuous map on a compact metric space and $\mu \in M(f)$ be ergodic. Let $G(\mu)$ be the set of generic points of μ, i.e.*

$$G(\mu) = \{x : V_f(x) = \{\mu\}\}.$$

Then $h(f, G(\mu)) = h_\mu(f)$.

Proposition 1.12. *If f and g are entropy conjugate homeomorphisms of compact metric spaces, then $h(f) = h(g)$.*

Proposition 1.13. *Suppose f and g are entropy- conjugate homeomorphisms of compact metric spaces. Then f is intrinsically ergodic iff g is.*

2 Recent Works

• In 2001 Oliver Jenkinson introduced the entropy for rotation vectors and relate this to the directional entropy of Geller and Misiurewicz.

Definition 2.1. *[6] Given a continuous function $g : X \to \mathbb{R}$ we define its pressure $P(g)$ with respect to T to be*

$$P(g) = sup_{\mu \in \mathcal{M}}(h(\mu) + \int g\, d\mu),$$

where $h(\mu)$ denotes the entropy of μ. If $m \in \mathcal{M}$ satisfies $P(g) = h(m) + \int g\, dm$ then it is called an equilibrium state for g.

Definition 2.2. *[6] Given $v \in \mathbb{R}^d$ we let $v.f$ denote the function $v_1 f_1 + ... + v_d f_d$. Define $p : \mathbb{R}^d \to \mathbb{R}$ by $p(v) = p(v.f)$. Of course p depends on both T and f, though for ease of notion we supress this dependence.*

Let $\mathcal{M}_f(v) = ES_{v.f}$ denote the set of equilibrium states of $v.f$, so that $\mathcal{M}_f$ is a map from $\mathbb{R}^d$ to the power set of $\mathcal{M}$. We will be interested in the d-parameter family $\mathcal{M}_f(\mathbb{R}^d)$. We call this the family of f-equilibrium states.

Definition 2.3. *[6] Let*

$$p'(v;h) = lim_{t\downarrow 0}\frac{p(v+th)-p(v)}{t}$$

denote the directional derivative of p at the point v in the direction $h \in \mathbb{R}^d$. A vector $u \in \mathbb{R}^d$ is a subgradient of p at the point $v \in \mathbb{R}^d$ if

$$p(v+h) \geq p(v) + u.h\, for\, all\, h \in \mathbb{R}^d.$$

The set of all subgradients of p at v is called the subdifferential of p at v, and is denoted by $\delta p(\mathbb{R}^d)$ denote $\cup_{v\in\mathbb{R}^d}\delta p(v)$.

Proposition 2.4. *[6] Let (X, T) be a dynamical system for which the entropy map is upper semi-continuous, and suppose $f : X \to \mathbb{R}^d$ is continuous. Then $\delta p(v) = f_*(\mathcal{M}_f(v))$ for all $v \in \mathbb{R}^d$.*

Corollary 2.5. *[6] Let (X, T) be a dynamical system for which the entropy map is upper semi-continuous, and suppose $f : X \to \mathbb{R}^d$ is continuous. Then p is differentiable at the point $v \in \mathbb{R}^d$ if and only if $f_*(\mathcal{M}_f(v))$ is a singleton.*

Proposition 2.6. *[6] Let (X,T) be a dynamical system for which the entropy map is upper semi continuous, and suppose $f: X \to \mathbb{R}^d$ is continuous. Then $f_*(\mathcal{M}) \subset \overline{\delta p(\mathbb{R}^d)}$.*

Theorem 2.7. *[6] Let (X,T) be a dynamical system for which the entropy map is upper semi-continuous, and suppose $f: X \to \mathbb{R}^d$ is continuous. Then*

$$f_*(\mathcal{M}) = \overline{f_*(\mathcal{M}_f(\mathbb{R}^d))} = \overline{\delta p(\mathbb{R}^d)}.$$

Corollary 2.8. *[6] Let (X,T) be a dynamical system for which the entropy map is upper semi-continuous, and suppose $f: X \to \mathbb{R}^d$ is continuous. Then $ri(f_*(\mathcal{M})) \subset f_*(\mathcal{M}_f(\mathbb{R}^d)) = \delta p(\mathbb{R}^d)$.*

Corollary 2.9. *[6] Let (X,T) be a dynamical system for which the entropy map is upper semi-continuous, and suppose $f: X \to \mathbb{R}^d$ is continuous. If p is strictly convex then $int(f_*(\mathcal{M})) = f_*(\mathcal{M}_f(\mathbb{R}^d))$.*

Lemma 2.10. *[6] If (X,T) is a mixing subshift of finite type, and $f: X \to \mathbb{R}^d$ is a cohomologically full function with summable variation. then*

(a) *Each $\mathcal{M}_f(v)$ contains a single measure, $m_{v.f}$ say.*

(b) *Each $m_{v.f}$ is fully supported.*

(c) *T has a Bernoulli natural extension with respect to $m_{v.f}$, and in particular the entropy $h(m_{v.f})$ is positive.*

(d) *If $v, v' \in \mathbb{R}^d$ with $v \neq v'$ then the equilibrium states $m_{v.f}, m_{v'.f}$ are distinct.*

(e) *The function p defined by $p(v) = P(v.f)$ is strictly convex.*

(f) *If f is a Holder continuous function then p is a real-analytic function of v.*

Corollary 2.11. *[6] Let (X,T) be a mixing subshift of finite type. Suppose $f: X \to \mathbb{R}^d$ is cohomologically full function with summable variation. Then $f_*(\mathcal{M}) \subset \mathbb{R}^d$ has interior, and $int(f_*(\mathcal{M})) = f_*(\mathcal{M}_f(\mathbb{R}^d))$.*

Definition 2.12. *[6] Given a dynamical system (X,T) and $g: X \to \mathbb{R}$ a continuous function, we say a measure $m \in \mathcal{M}$ is g-optimal if $\int g\,dm = sup_{\mu\in\mathcal{M}} \int g\,d\mu$. We call $Q(g) = sup_{\mu\in\mathcal{M}} \int g\,d\mu$ the optimal ergodic average for g.*

Lemma 2.13. *[6] Let* (X,T) *be a subshift of finite type. Let* $g : X \to \mathbb{R}$ *be summable variation. A measure* $m \in \mathcal{M}$ *is* g*-optimal if and only if there exits* $\phi \in C(X)$ *such that*

$$\phi(Tz) + Q(g) = \phi(z) + g(z)\ for\ all\ z \in supp(m).$$

Furthermore, if X *is one-sided and* $g \in \mathscr{F}_\theta(X)$ *then any such that* ϕ *also belongs to* $\mathscr{F}_\theta(X)$*. If* X *is two-sided and* $g \in \mathscr{F}_\theta(X)$ *then any such* ϕ *belongs to* $\mathscr{F}_{\sqrt{\theta}}(X)$*.*

Corollary 2.14. *[6] Let* (X,T) *be a subshift of finite type, and* $g : X \to \mathbb{R}$ *be of summable variation. Suppose* $\mu \in \mathcal{M}$ *is* g*-optimal, and* $m \in \mathcal{M}$ *satisfies* $supp(m) \subset supp(\mu)$*. Then* m *is also* g*-optimal.*

Corollary 2.15. *[6] Let* (X,T) *be a subshift of finite type, and* $g : X \to \mathbb{R}$ *be of summable variation. If there is a unique* g*-optimal measure* $\mu \in \mathcal{M}$*, then the restriction of* T *to* $supp(\mu)$ *is uniquely ergodic.*

Definition 2.16. *[6] Given a dynamical system* (X,T) *we define the entropy function* H *on* $f_*(\mathcal{M}$ *by*

$$H(e) = sup\{h(\mu) : f_*(\mu) = e\}.$$

Theorem 2.17. *[6] Let* (X,T) *be a dynamical system for which the entopy map is upper semi-continuous, and suppose* $f : X \to \mathbb{R}^d$ *is continuous. Let* $e \in ri(f_*(\mathcal{M}))$*.*

(a) *There exists* $v \in \mathbb{R}^d$ *for which* $f^{-1}(e) \cap \mathcal{M}_f(\mathbb{R}^d) \subset \mathcal{M}_f(v)$*.*

(b) *A measure* $m \in f^{-1}(e)$ *satisfies* $h(m) = H(e)$ *if and only if* $m \in f^{-1}(e) \cap \mathcal{M}_f(\mathbb{R}^d)$*.*

Theorem 2.18. *[6] Let* (X,T) *be a mixing subshift of finite type, and suppose* $f : X \to \mathbb{R}^d$ *is cohomologically full and has summable variation. Let* $e \in int(f_*(\mathcal{M}))$*. Then*

(a) $f^{-1}(e)$ *intersects* $\mathcal{M}_f(\mathbb{R}^d)$ *at a single measure* m*.*

(b) m *is the unique measure in* $f_*^{-1}(e)$ *satisfying* $h(m) = H(e)$*.*

Definition 2.19. *[6] An ergodic measure* $\mu \in \mathcal{M}$ *is called directional if* $f_*(m) = f_*(\mu)$ *for all* $m \in \mathcal{M}$ *satisfying* $supp(m) \subset supp(\mu)$*. An ergodic measure* $\mu \in \mathcal{M}$ *is called lost if it is not directional.*

Proposition 2.20. *[6] Let (X, T) be a dynamical system with upper semi-continuous entropy map, and suppose $f : X \to \mathbb{R}^d$ is continuous. Suppose $\mu \in \mathcal{M}$. Then μ is directional if either of the two following condition hold*

(a) *for all i, the restricted coordinate function $f_i|_{supp(\mu)}$ is an essential coboundary for the dynamical system $(supp(\mu), T|_{supp(\mu)})$,*

(b) *the restriction of T to $supp(\mu)$ is uniquely ergodic.*

Proposition 2.21. *[6] Let (X, T) be a mixing subshift of finite type, and $f : X \to \mathbb{R}^d$ be of summable variation, with at least one of its coordinate functions not an essential coboundary. Then*

(a) *Any fully supported ergodic measure $\mu \in \mathcal{M}$ is lost,*

(b) *Any f-equilibrium state $m_{v.f} \in \mathcal{M}_f(\mathbb{R}^d)$ is lost.*

Proposition 2.22. *[6] Let (X, T) be a subshift of finite type, $f : X \to \mathbb{R}^d$ be of summable variation, and e an exposed point of the rotation set $f_*(\mathcal{M})$. Then every ergodic measure in the rotation class $f_*^{-1}(e)$ is directional, and $f_*^{-1}(e)$ contains at least one ergodic measure.*

Definition 2.23. *For $e \in f_*(\mathcal{M})$ the directional entropy $\mathcal{H}(e)$ at the point e is given by*

$$\mathcal{H}(e) = sup\{h(\mu) : \mu \in f_*^{-1}(e)\, is\, directional\},$$

Where we define the supremum of the empty set to be 0.

Theorem 2.24. *[6] Let (X, T) be a subshift of finite type, and suppose $f : X \to \mathbb{R}^d$ has summable variation. If e is an exposed point of rotation set then there exists at least one directional measure $m \in f_*^{-1}(e)$ with $h(m) = H(e)$. Consequently the directional entropy $\mathcal{H}(e)$ is equal to entropy $H(e)$ at all exposed points e of $f_*(\mathcal{M})$.*

Lemma 2.25. *[6] Let X be a mixing subshift of finite type, with alphabet $\{1, ..., k\}$. Let $f : X \to \mathbb{R}^d$ be continuous. For $\epsilon > 0, a \in \{1, ..., k\}$, and for arbitrary large $M \in \mathbb{N}$, there exists a finite collection C of length M periodic blocks, each begining with symbol a, whose rotation vectors are ϵ-dense in $f_*(\mathcal{M})$.*

Lemma 2.26. *[6] Let X be a mixing subshift of finite type, with alphabet $1,...,k$. Let $f : X \to \mathbb{R}^d$ be continuous. For any $a \in \{1,...,k\}, e \in f_*(\mathcal{M}), r > 0, n \in \mathbb{N}$, let $M(e,r,n)$ be the number of length n-periodic blocks begining with symbol a whose rotation vectors lie in $B_r(e)$. Then*

$$H(e) = lim_{r\to 0} limsup_{n\to\infty} \frac{1}{n} logM(e,r,n).$$

Theorem 2.27. *[6] Let X be a mixing subshift of finite type. Let $f : X \to \mathbb{R}^d$ be of summable variation, and cohomologically full. For $e \in int(f_*(\mathcal{M}))$ we have that $\mathcal{H}(e) = H(e)$. That is, the directional entropy $\mathcal{H}$ coincides with the entropy function H on the interior of the rotation set.*

Proposition 2.28. *[6] Let (X,T) be a mixing subshift of finite type, and $f : X \to \mathbb{R}^d$ be of summable variation and cohomologically full. If e is in the interior of the rotation set $f_*(\mathcal{M})$ then there exists unique measure $m \in f^{-1}(e)$ with $h(m) = H(e) = \mathcal{H}(e)$, and this measure is lost.*

Theorem 2.29. *[6] Let (X,T) be a mixing subshift of finite type, and suppose $f : X \to \mathbb{R}^d$ is cohomologically full and has summable variation. Then $\mathcal{H}(e) = H(e)$ at all interior and exposed point e of he rotation set $f_*(\mathcal{M})$. For e in the interior of $f_*(\mathcal{M}), \mathcal{H}(e) = H(e)$ is attained by a unique measure in the rotation class $f_*^{-1}(e)$, and this measure is lost. If e is an exposed point of $f_*(\mathcal{M})$ then $\mathcal{H}(e) = H(e)$ is attained by at least one directional measure in $f_*^{-1}(e)$, and is not attained by any lost measure in $f_*^{-1}(e)$.*

Corollary 2.30. *[6] Let (X,T) be a mixing subshift of finite type, and suppose $f : X \to \mathbb{R}^d$ is cohomologically full with summable variation. If the rotation set $f_*(\mathcal{M})$ is strictly convex, then the functions $\mathcal{H}$ and H coincide.*

Lemma 2.31. *[6] Suppose (X,T) is a transitive subshift of finite type, and that $f : X \to \mathbb{R}^d$ is constant on cylinders of length two. Let $\mu_1,...,\mu_r$ be the invariant measure supported on the elementary periodic orbits of X. Then the rotation set $f_*(\mathcal{M})$ is a polyhedron whose extermal points are a subset of $f_*(\mu_1),...,f_*(\mu_r)$.*

Corollary 2.32. *[6] Suppose (X,T) is a transitive subshift of finite type, and that $f : X \to \mathbb{R}^d$ is locally constant function. Then the rotation set $f_*(\mathcal{M})$ is a polyhydron.*

Theorem 2.33. *[6] Suppose (X,T) is a transitive subshift of finite type, and that $f : X \to \mathbb{R}^d$ is constant on cylinders of length two. Let F be a face of corresponding polyhedral* *rotation set $f_*(\mathcal{M})$, and $L_1,...,L_s$ those elementary loops of X whose rotation vectors $e_1,...,e_s$ lie in F. If $X_F \subset X$ denotes the non-wandering subshift of finite type generated by loops $L_1,...,L_s$, then for any measure $\mu \in \mathcal{M}$,*

$f_*(\mu) \in F$ if and only if $supp(\mu) \subset X_F$.

• In 2012 De-Jun Feng and Wen Huang defined the measure-theoretical lower and upper entropies $\underline{h}_\mu(T), \overline{h}_\mu(T)$ for any $\mu \in M(X)$, where $M(X)$ denotes the collection of all Borel probability measures on X. For any non-empty compact subset K of X, they showed that

$$h^B_{top}(T,K) = sup\{\underline{h}_\mu(T) : \mu \in M(X),\ \mu(K) = 1\},$$

$$h^P_{top}(T,K) = sup\{\overline{h}_\mu(T) : \mu \in M(X),\ \mu(K) = 1\},$$

Where $h^B_{top}(T,K)$ denotes the Bowen topological entropy of K, and $h^P_{top}(T,K)$ the packing topological entropy of K.

Definition 2.34. *[18] Let $\mu \in M(X)$. The measure theoretical lower and upper entropies of μ are defined respectively by*

$$\underline{h}_\mu(T) = \int \underline{h}_\mu(T,x)d\mu(x),\ \overline{h}_\mu(T) = \int \overline{h}_\mu(T,x)d\mu(x),$$

where

$$\underline{h}_\mu(T,x) = lim_{\epsilon\to 0} liminf_{n\to+\infty} -\frac{1}{n} log\mu(B_n(x,\epsilon)),$$

$$\overline{h}_\mu(T,x) = lim_{\epsilon\to 0} limsup_{n\to+\infty} -\frac{1}{n} log\mu(B_n(x,\epsilon)).$$

Theorem 2.35. *[18] Let (X,T) be a TDS.*

(a) *If $K \subseteq X$ is non-empty and compact, then*

$$h^B_{top}(T,K) = sup\{\overline{h}_\mu(T) : \mu \in M(X), \mu(K) = 1\}.$$

(b) *Assume that $h_{top}(T) < \infty$. If $Z \subseteq X$ is analytic, then*

$$h^B_{top}(T,Z) = sup\{h^B_{top}(T,K) : K \subseteq Z\, is\, compact.\}.$$

Theorem 2.36. *[18] Let (X,T) be a TDS.*

(a) *If $K \subseteq X$ is non-empty and compact, then*

$$h_{top}^{P}(T,K) = sup\{\overline{h}_{\mu}(T) : \mu \in M(X), \mu(K) = 1\},$$

where $h_{top}^{P}(T,K)$ denotes the packing topological entropy of K.

(b) *If $Z \subseteq X$ is analytic, then*

$$h_{top}^{P}(T,Z) = \{h_{top}^{P}(T,K) : K \subseteq Z \, is \, compact\}.$$

Proposition 2.37. *[18]*

(a) *For $Z \subseteq Z'$,*

$$h_{top}^{UC}(T,Z) \le h_{top}^{UC}(T,Z'),\ h_{top}^{B}(T,Z) \le h_{top}^{B}(T,z'),\ h_{top}^{P}(T,Z) \le h_{top}^{P}(T,Z').$$

(b) *For $Z \subseteq \cup_{i=1}^{\infty} Z_i$, $s \ge 0$ and $\epsilon > 0$, we have*

$$\mathscr{M}_{\epsilon}^{s}(Z) \le \Sigma_{i=1}^{\infty} \mathscr{M}_{\epsilon}^{s}(Z_i),$$

$$c \le sup_{i\ge1} h_{top}^{B}(T,Z_i),\ h_{top}^{P}(T,Z) \le sup_{i\ge1} h_{top}^{P}(T,Z_i).$$

(c) *For any $Z \subseteq X$, $h_{top}^{B}(T,Z) \le h_{top}^{P}(T,Z) \le h_{top}^{UC}(T,Z)$.*

(d) *Furthermore, if Z is T-invariant and compact, then*

$$h_{top}^{B}(T,Z) = h_{top}^{P}(T,Z) = h_{top}^{UC}(T,Z).$$

Proposition 2.38. *[18]*

(a) *For any $s \ge 0, N \in \mathbb{N}$ and $\epsilon > 0$ both $\mathscr{M}_{N,\epsilon}^{s}$ and $\mathscr{W}_{N,\epsilon}^{s}$ are outer measures on X.*

(b) *For any $s \ge 0$, both $\mathscr{M}^{s}$ and $\mathscr{W}^{s}$ are metric outer measures on X.*

Proposition 2.39. *[18] Let $Z \subseteq X$. Then For any $s \ge 0$ and $\epsilon, \delta > 0$, we have*

$$\mathscr{M}_{N,6\epsilon}^{s+\delta}(Z) \le \mathscr{W}_{N,\epsilon}^{s}(Z) \le \mathscr{M}_{N,\epsilon}^{s}(Z)$$

when N is large enough. As a result, $\mathscr{M}^{s+\delta}(Z) \le \mathscr{W}^{s}(Z) \le \mathscr{M}^{s}(Z)$ and $h^{B}_{top}(T,Z) = h^{WB}_{top}(T,Z)$.

Lemma 2.40. *[18] Let (X,d) be a compact metric space and $\mathscr{B} = \{B(x_i,r_i)\}_{i\in\mathscr{T}}$ be a family of closed balls in X. Then there exists a finite or countable subfamily $\mathscr{B}'\{B(x_i,r_i)\}_{i\in} i \in \mathscr{T}'$ of pairwise disjoint balls in $\mathscr{B}$ such that*

$$\cup_{B\in\mathscr{B}} B \subseteq \cup_{i\in\mathscr{T}} B(x_i, 5r_i).$$

Lemma 2.41. *[18] Let K be a non-empty compact subset of X. Let $s \ge 0, N \in \mathbb{N}$ and $\epsilon > 0$. Suppose that $c := \mathscr{W}_{N,\epsilon} s(K) > 0$. Then there is a Borel measure μ on X such that $\mu(K) = 1$ and*

$$\mu(B_n(x,\epsilon)) \le \frac{1}{c} e^{-ns}, \forall x \in X, n \ge N.$$

Theorem 2.42. *[18] Let (X,T) be a TDS. Assume that X is zero-dimensional, i.e., for any $\delta > 0$, X has a closed-open partition with diameter less than δ. Then for any analytic set $Z \subset X$,*

$$h^{B}_{top}(T,Z) = sup\{h^{B}_{top}(T,K) : K \subset Z, K\, is\, compact\}.$$

Proposition 2.43. *[18] Assume $\mathscr{U}$ is closed-open partition of X. Let $N \in \mathbb{N}$. Then*

(a) *If $E_i \uparrow E$, i.e., $E_{i+1} \supseteq E_i$ and $\cup_i E_i = E$, then*

$$\mathscr{M}^{s}_{N}(\mathscr{U},E) = lim_{i\to\infty} \mathscr{M}^{s}_{N}(\mathscr{U},E_i).$$

(b) *Assume $Z \subset X$ is analytic. Then*

$$\mathscr{M}^{s}_{N}(\mathscr{U},Z) = sup\{\mathscr{M}^{s}_{N}(\mathscr{U},K) : K \subset Z, K\, is\, compact\}.$$

Lemma 2.44. *[18] Let (X,T) be a TDS with metric d and a surjective map T, $(\tilde{X},\tilde{T})$ be the natural extension of (X,T) and $\pi_1 : \tilde{X} \to X$ be the projection of the first coordinate. Then $sup_{x\in X} h^{UC}_{top}(\tilde{T}, \pi_1^{-1}(x)) = 0$.*

Definition 2.45. *An extension $\pi : (Z,R) \to (X,T)$ between two TDS is a principal extension if $h_\nu(R) = h_{\nu\circ\pi^{-1}}(T)$ for every $\nu \in M(Z,R)$.*

Proposition 2.46. *[18] Every invertible TDS (X,T) with $h_{top}(T)<\infty$ has a zero-dimensional principal extension (Z,R) with R being invertible.*

Theorem 2.47. *Let $\pi:(Y,S)\to(X,T)$ be a factor map between two TDSs. Then for any $E\subseteq Y$ one has*

$$h^B_{top}(T,\pi(E))\le h^B_{top}(S,E)\le h^B_{top}(T,\pi(E))+sup_{x\in X}h^{UC}_{top}(S,\pi^{-1}(x)).$$

Proposition 2.48. *[18] Let $\pi:(Y,S)\to(X,T)$ be a factor map between two TDSs with $h_{top}(S)<\infty$. Then we have*

$$sup_{x\in X}h^{UC}_{top}(S,\pi^{-1}(x))=sup_{\mu\in M(Y,S)}(h_\mu(S)-h_{\mu\circ\pi^{-1}}(T).$$

Lemma 2.49. *[18] Let (X,T) be a TDS with $h_{top}(S)<\infty$.Then there exists a factor map $\pi:(H,\Gamma)\to(X,T)$ such that (H,Γ) is zero-dimesional and*

$$sup_{x\in X}h^{UC}_{top}(\Gamma,\pi^{-1}(x))=0.$$

Lemma 2.50. *Let $Z\subset X$ and $s,\epsilon>0$. Assume $P^s_\epsilon(Z)=\infty$. Then for any given finite interval $(a,b)\subset\mathbb{R}$ with $a\ge0$ and any $N\in\mathbb{N}$, there exists a finite disjoint colection $\{\overline{B}_{n_i}(x_i,\epsilon)\}$ such that $x_i\in Z$, $n_i\ge N$ and $\Sigma_i e^{-n_i s}\in(a,b)$.*

• In August, 2020 Xiankun Ren, Xueting Tian and Yunuha Zhou proved a variational principle for topological entropy of saturated sets for systems which have specification and uniform separation properties. They worked on group action topological system, where the group is a countable infinite discrete amenable group with a compact metric space.

Theorem 2.51. *[8] If the specification and uniform separation property hold, then for any non-empty connected closed subset $K\subset M(X,G)$*

$$h^B_{top}(G_K,\{F_n\})=inf\{h_\mu(X,G)|\mu\in K\}.$$

Theorem 2.52. *v Suppose the system (X,G) has the specification and uniform seperation properties. If $\hat{X}(\phi,\{F_n\})$ is non-empty then*

$$h^B_{top}(X(\phi,\alpha.\{F_n\}))=h_{top}(X,G).$$

Theorem 2.53. *[8] For* $\alpha \in \mathbb{R}$,

$$h^B_{top}(X(\phi,\alpha.\{F_n\})) = sup\{h_\mu(X,G), \int_X \phi d\mu = \alpha\}.$$

Definition 2.54. *[8] Let* $\Omega, K \subset F(G)$ *be two subsets of a group* G*. The* $K-interior$ *of* Ω *is the subset* $Int_K(\Omega)$ *defined by*

$$Int_K(\Omega) := \{g \in G | Kg \subset \Omega\}.$$

The $K-closure$ *of* Ω *is the subset* $Cl_K(\Omega) \subset G$ *defined by*

$$Cl_K(\Omega) := \{g \in G | Kg \cap \Omega \neq \phi\}.$$

The $K-boundary$ *of* Ω *is the subset* $\partial_K(\Omega) \subset G$ *defined by*

$$\partial_K(\Omega) := Cl_K(\Omega)\ Int_K(\Omega).$$

The relative amenability constant of Ω *with respect to* K *is the number* $\alpha(\Omega,K)$ *defined by*

$$\alpha(\Omega,K) := \frac{|\partial_K(\Omega)}{|\Omega|}.$$

Ω *is called* $(K,\delta)-invariant$ *if* $\alpha(\Omega,K) < \delta$.

Lemma 2.55. *[8] Let* G *be a countable amenable group. Let* $\{F_n\}$ *be a Folner sequence. For any finite subset* $K \subset G$,

$$lim_{n\to\infty}\alpha(F_n,K) = 0$$

Proposition 2.56. *[8] Let* G *be a group and* $0 < \epsilon \leq \frac{1}{2}$*. Then there exists an integer* $s_0 = s_0(\epsilon) \geq 1$ *such that for each* $s \leq s_0$ *the following holds, If* $K_1, K_2, ..., K_s$ *are non-empty finite subset of* G *such that*

$$\alpha(D,K_j) \leq \epsilon^{2\epsilon}\ for\ all\ 1 \leq j < k \leq s,$$

and D *is a non-empty finite subset of* G *such that*

$$\alpha(D,K_j) \leq \epsilon^{2\epsilon}\ for\ all\ 1 \leq j \leq s,$$

then D *can be* $\epsilon - quasi$ *tiled by* $K_1, ...K_s$.

Definition 2.57. *[8] $\mathscr{T}$ is called a tiling of G if there exists a shape set $\mathscr{S} = \{S_i \in F(G) | 1 \leq i \leq k\}$ and tiling centers $C_1, C_2, ..., C_k$ such that $\{S_j g | g \in C_j, j = 1, 2, ..., k\}$ ia a partition of G. Let $\{\mathscr{T}_k\}_{k\geq 1}$ be a sequence of tilings of G, we say $\{\mathscr{T}_k\}_{k\geq 1}$ is congruent if for each $k \geq 1$, each element in $\{\mathscr{T}_{k+1}\}$ is a union of elements in $\{\mathscr{T}_k\}$.*

Lemma 2.58. *[8] Fix a converging to zero sequence $\epsilon_k > 0$ and a sequence K_k of finite subsets of G. There exists a congruent sequence of tilings $\mathscr{T}_k$ of G such that shapes of $\tilde{T}_k$ are $(K_k, \epsilon_k) - invariant$.*

Lemma 2.59. *[8] Let (X, G) be a dynamical system. Let $\mu \in M(X, G)$, $\delta^* > 0, \epsilon^* > 0, \xi > 0$. Then there exists $0 < \delta < min\{\frac{1}{2}, \frac{\xi}{3}, \frac{\delta^*}{2}\}$ such that if $F \in F(G)$ and $\Gamma \subset X_{F, \mathscr{B}(\mu,\xi)}$ is a (δ^*, F, ϵ)-seperated set, then for any $F' \subset F$ with $\frac{|F'|}{|F|} > 1 - \delta$, Γ is a (δ^*, F, ϵ)-seperated set, then for any $F' \subset F$ with $\frac{|F'|}{|F|} > 1 - \delta, \Gamma$ is a $(\frac{\delta^*}{2}, F', \epsilon^*)$ set and $\Gamma \subset X_{F', \mathscr{B}(\mu, 2\epsilon)}$.*

Definition 2.60. *[8] The measure $v \in M(X, G)$ is entropy-approachable by ergodic measure if for any neighbourhood C of v and each $h^* < h_v(X, G)$, there exists a measure $u \in E(X, G) \cap C$ such that $h_\mu(X, G) > h^*$. The ergodic measures are entropy dense if each $v \in M(X, G)$ is entropy-approachable by ergodic measures.*

Theorem 2.61. *[8] Suppose the dynamical system has specification property. Then the ergodic measures are entropy dense.*

Lemma 2.62. *[8] If $\binom{n}{k}$ denotes the number of combinations of n objects taken k at a time and $\delta < 1/2$ then*

$$\Sigma_{k\leq \delta n} \begin{pmatrix} n \\ k \end{pmatrix} \leq e^{n\phi(\delta)},$$

where $\phi(\delta) = -\delta log\delta - (1 - \delta)(log(1 - \delta))$.

Theorem 2.63. *[8] Suppose the action (X, G) is expensive. Then the action has the uniform separation property.*

Theorem 2.64. *[8] Let Γ be a countable discrete group and f an element of $\mathbb{Z}\Gamma$ invertible in $l^1(\Gamma, \mathbb{R})$. Then the action of Γ on X_f which is the Pontryagin dual of $\mathbb{Z}\Gamma/\mathbb{Z}\Gamma f$ has the specification and uniform separation properties.*

Corollary 2.65. *[8] Assume that (X, ρ, G) has the uniform seperation property and that the ergodic measures are entropy dense. Let $\{K_n\}$ be a tempered Folner sequence. For* *any $\eta > 0$, there exists $\delta^* > 0$ and $\epsilon^* > 0$ so that for $\mu \in M(X,G)$ and any neighborhood $C \subset M(X)$ of μ, there exists $n^*_{C,\mu,\eta}$ such that*

$$N(C;\delta^*, K_{n,\epsilon^*}) \geq e^{|K_n|(h_\mu(X,G)-\eta)}$$

For any $\mu \in M(X,G)$,

$$h_\mu(X,G) \leq lim_{\epsilon\to 0} lim_{\delta\to 0} inf_{C\ni\mu} \frac{1}{|K_n|} log N(C;\delta,K_n,\epsilon).$$

Proposition 2.66. *[8] Let (X,G) be a topological dynamical system and $\mu \in M(X,G)$. Then for any Folner sequence $\{F_n\}$,*

$$\overline{s}(\mu,\{F_n\}) \leq h_\mu(X,G).$$

Proposition 2.67. *[8] Let $\{K_n\}$ be a tempered Folner sequence and $\mu \in E(X,G)$. Then for $h^* < h_\mu(X,G)$, there exist $\delta^* > 0, \epsilon^* > 0$ such that for any neighborhood C of μ, there exists n^*_C, s.t for any $n \geq n^*_C$ there exists a $(\delta^*, F_n, \epsilon^*)$-separated set Γ_n of $X_{F_n,C}$ satisfying*

$$|\Gamma_n| \geq e^{h|K_n|}.$$

Corollary 2.68. *[8] Let (X,G) be a topological dynamical system, $\{K_n\}$ be a tempered Folner seqence. For $\mu \in E(X,G)$,*

$$h_\mu(X,G) = h_\mu(X,\{F_n\}) = lim_{\epsilon\to 0} lim_{\delta\to 0} \underline{s}(\nu;\delta,\epsilon,\{K_n\})$$

$$= lim_{\epsilon\to 0} lim_{\delta\to 0} \overline{s}(\nu;\delta,\epsilon,\{K_n\}).$$

Proposition 2.69. *[8] Let (X,G) be a topological dynamical system. If the uniform separation property condition is true and the ergodic measures are entropy dense, then for a tempered Folner sequence $\{K_n\}$, $s(\mu,\{K_n\})$ is well-defined, and $s(\mu,\{K_n\}) = h_\mu(X,\{K_n\}) = h_\mu(X,G)$, for all $\mu \in M(X,G)$.*

Proposition 2.70. *[8] Let (X,G) be a topological dynamical system. Let $\{K_n\}$ be a tempered Folner sequence. If the uniform seperation property condition is true and the er-*

godic measures are entropy dense, then the entropy map

$$\mu \to s(\mu; \{F_n\})$$

is upper semi-continuous.

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A study on the method of Rothe for solving fractional integral and differential equations

Darshana Devi
Department of Mathematical Sciences, Tezpur University, Sonitpur, Assam, India

email: darsana.mou@gmail.com

Abstract. This article is a study on the Method of Rothe. The aim of this article is to present Rothe's method (also called method of semidiscretization, or the method of lines) as an effective tool for solving a fractional integral and differential equations. As an application, we provide the work done by A. Raheem - D. Bahuguna ([2], [4]) to discuss the use of Rothe Method. The main objective of this write-up is to inspire interested readers to go through the works described here.

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1 Introduction

Rothe's method was introduced by E. Rothe [1] in 1930 for the purpose of solving the following scalar parabolic initial boundary value problem of second order,

$$R(t,x)\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2}=S(t,x,u),\qquad 0<x<1,\,t>0,$$
$$u(0,x)=u_0(x),$$
$$u(t,0)=u(t,1)=0,\qquad t\geq 0,$$

where R and S are sufficiently smooth functions in the variables t and x in $[0,T]\times(0,1)$. Here T means an arbitrary finite positive number. Rothe's method consists in dividing the interval $[0,T]$ into n number of subintervals of the form $[t^n_{j-1},t^n_j]$, $t^n_j=jh$, $j=1,2,\ldots,n$ with $t^n_0=0$, of equal lengths $h(=\frac{T}{n})$ and replacing the partial derivative $\frac{\partial u}{\partial t}$ of the unknown function u by the difference quotients $\frac{u^n_j-u^n_{j-1}}{h}$. After defining a sequence of polygonal functions as,

$$U^n(x,t)=u^n_{j-1}(x)+\frac{1}{h}(t-t^n_{j-1})(u^n_j(x)-u^n_{j-1}(x)),\qquad t\in[t^n_{j-1},t^n_j],$$

Rothe proved that the sequence $\{U^n\}$ converges to the unique solution of the problem as $n\to\infty$ using some apriori estimates on the sequence $\{U^n\}$. The method introduced by Rothe becomes a very effective theoretical tool for proving the existence and uniqueness of solutions of linear, nonlinear, parabolic and hyperbolic problems of higher orders. This method is presently known as *Rothe's method.* It is also known as *the method of semidiscretization* or *the method of lines.* This method can be used in diffusion problems also as e.g. [2, 3, 5, 6, 12]. Thus Rothe's method has many applications not only in mathematics but also in physical and biological problems designed by partial differential equations.

As an application of Rothe's method to a semilinear differential equation readers can see the section 5.

For the use of Rothe's method to show the existence and uniqueness of a strong solution of a quasilinear equation readers can follow [17], in which the author consider the problem

X and Y be two real reflexive Banach spaces such that Y is densely and continuously embedded in X. Consider the following integrodifferential equation in X

$$\frac{du(t)}{dt} + A(u(t))u(t) = K(u)(t) + f(t), \quad 0 < t < T, u(0) = u_0,$$

where $A(u)$ is a linear operator in X for each u in an open subset W of Y, K is the nonlinear Volterra operator

$$K(u)(t) = \int_0^t a(t-s)k(s,u(s))ds,$$

where a is a real-valued function defined on $J := [0,T]$ and k is the Y-valued map defined on $J \times W$, $f : J \to Y$.

For a nonlinear problem readers can see [11].
In [11] using the method of semidiscretization author has established the existence and uniqueness of a strong solution for the following nonlinear nonlocal functional differential equation in a Banach X

$$u'(t) + Au(t) = f(t, u(t), u_t), \quad t \in (0,T],$$
$$h(u_0) = \phi \quad on[-\tau, 0],$$

where $0 < T < \infty$, $\phi \in C_0 := C([-\tau, 0]; X)$, $\tau > 0$, the nonlinear operator A is singlevalued and m-accretive defined from the domain $D(A) \subset X$ into X, the nonlinear map f is defined from $[0,T] \times X \times C_0 := C([-\tau,0]; X)$ into X, the map h is defined from C_0 into C_0. For $u \in C_\tau := C([-\tau, T] : X)$, function $u_t \in C_0$ is given by $u_t(s) = u(t+s)$ for $s \in [-\tau, 0]$. Here $C_t := C([-\tau, t]; X)$ for $t \in [0,T]$ is the Banach space of all continuous functions from $[-\tau, t]$ into X endowed with the supremum norm

$$\|\phi\|_t = sup_{-\tau \leq \eta \leq t}, \quad \phi \in C_t.$$

Later on many authors have applied and developed the Rothe's Method to various types of differential equations, we refer the readers to [8, 10, 11, 13, 14].

As our main aim is to present the Method of Rothe as an effective tool for solving fractional integral and differential equation, so before going to discuss the work of Raheen - Bahuguna ([2], [4]) we state some definitions and results related to the mentioned work.

2 Fractional Calculus

The first appearance of fractional calculus was in a letter exchange between Leibniz and a French Mathematician Marquis de L'Hospital in 1695. For n^{th} order derivative where $n \in \mathbb{N}$ Leibniz introduced the notation $\frac{d^n y}{dx^n} = D^n y$ and repotted this to L'Hospital. In his letter L'Hospital asked the question to Leibniz, "What if $n = \frac{1}{2}$?" (i.e. if n is fractional). Leibniz replied, "......$d^{\frac{1}{2}}x$ will be equal to $x\sqrt{dx : x}$. This is an apparent paradox from which, one day, useful consequences will be drawn. Since there are little paradoxes without usefulnes.....". That was the first step of fractional calculus. After that it drew attention of many mathematicians such as Abel, Laplace, Euler, Riemann, Fourier and so on.

The fractional calculus has received a significant attention in the recent years due to its physical background in the field of engineering, physics, mathematics, chemistry, economics etc. It is a powerful tool which plays an major role in the study of nonlinear oscillations of earthquakes and the modeling of multiscale problems. Differential equations with fractional order derivative or fractional integration are found to be more suitable in comparison to the integer order derivative or integration in providing a mathematical aspect of physical phenomena.

From a numerical perspective, different understandings of fractional differentiation have been proposed, however there is as yet a profound discussion. The fractional differentiation and integration of nonlocal operators are not yet all around characterized and that different definitions are still exist together. From the main reference work recorded in 1695 to the current day, numerous articles have been published on this topic, yet there is still a ton to do.

The commonly used definitions of fractional differentiation and integration are given by Riemann, Liouville and Caputo. We define Riemann-Liouville fractional derivatives and integrals and Caputo fractional derivatives in the following.

Definition 2.1. *(Riemann-Liouville fractional integration) Let $f \in L_1[a,b], \alpha > 0$ the Riemann-Liouville fractional integral operator of order α is defined as*

$$I_a^\alpha f(t)[or D_a^{-\alpha} f(t)] = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds \quad \forall t \in [a,b].$$

Definition 2.2. *(The Riemann-Liouville fractional derivative) The Riemann-Liouville* *fractional derivative of $f(t)$ of order α on a finite interval $[a,b]$ is defined as*

$$D_a^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-s)^{n-\alpha-1} f(s)ds, \qquad n-1 \le \alpha < n.$$

Definition 2.3. *(Caputo fractional derivatives) Let $n-1 \le \alpha < n, n \in \mathbb{Z}^+$, the Caputo fractional derivative of a function $f(t)$ of order α is defined by*

$${}^cD_a^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f^n(s)ds.$$

Remark 2.4. *The Euler's gamma function is defined as,*

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt.$$

In [15] authors Lin-Xu used method based on time discretization to the following time fractional diffusion problem

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} - \frac{\partial^2 u(x,t)}{\partial x^2} = f(x,t), \qquad x \in \wedge, 0 < t \le T.$$

Subject to the following initial and boundary conditions

$$u(x,0) = g(x), \qquad x \in \wedge,$$
$$u(0,t) = u(L,t) = 0, \qquad 0 \le t \le T,$$

where $0 < \alpha < 1$ is the order of the time fractional derivative. The term $\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}$ is defined as Caputo fractional derivative of order α, given by

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,s)}{\partial s} \frac{ds}{(t-s)^\alpha}.$$

In [16], authors Sweilam-Khader-Mahdy used the Crank–Nicolson finite difference method to solve the following linear time-fractional diffusion equation with Dirichlet boundary conditions

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^2 u(x,t)}{\partial x^2},$$
$$u(x,0) = f(x),$$
$$u(0,t) = u(1,t) = 0,$$

where $0 < x < 1, 0 \le t \le T$ and the parameter $0 < \alpha < 1$ refers to the fractional order of the time derivative.

Recently, the existence and uniqueness of a strong solution for the following multi-term time fractional integral diffusion equation have been established by Migórski - Zeng [5] in a Hilbert space H applying Rothe's Method,

$$\frac{\partial u(t)}{\partial t} + Au(t) = \sum_{i=1}^{k} a_i({}_0I_t^{\alpha_i} u(t)) + f(t), \qquad t \in (0,T],$$
$$u(0) = u_0.$$

where the constants $a_i, \alpha_i, i = 1, ..., k$ are such that $a_i \ge 0, \alpha_i > 0, -A : D(A) \subset H \to H$ is an infinitesimal generator of a C_0-semigroup of contractions in H, the function $f : [0,T] \to H$ is Lipschitz continuous, ${}_0I_t^{\alpha_i} u$ denotes the fractional integral of order $\alpha_i > 0$ of u, and $u_0 \in D(A)$.

Readers can check [18] for the time discretization in fractional differential equations.

Besides this many author have done different work using fractional differential equation such as [2, 4, 6].

3 Semigroup Theory

3.1 Semigroup of bounded linear operators

Definition 3.1. *A one parameter family $T(t)$, $0 \le t < \infty$, of bounded linear operators from a Banach space X into X is a semigroup of bounded linear operators on X if*

(i) $T(0) = I$, (I is the identity operator on X),

(ii) $T(t+s) = T(t)T(s)$ *for every* $t, s \geq 0$ *(the semigroup property).*

Definition 3.2. *A semigroup* $T(t), 0 \leq t < \infty$*, of bounded linear operators on* X *is a strongly continuous semigroup of bounded linear operators if*

$$\lim_{t \downarrow 0} T(t)x = x \text{ for every } x \in X.$$

A strongly continuous semigroup of bounded linear operators on X *is also known as semigroup of class* C_0 *or simply* C_0 *semigroup.*

Definition 3.3. *Let* $T(t), 0 \leq t < \infty$ *be a semigroup of bounded linear operators on* X*. The infinitesimal generator* $A : D(A) \subset X \to X$ *of* $T(t)$ *is defined by*

$$Ax = \lim_{t \downarrow 0} \frac{T(t)x - x}{t}, \quad x \in D(A),$$

where $D(A)$ *is the domain of* A *and it is defined as*

$$D(A) = \left\{ x \in X : \lim_{t \downarrow 0} \frac{T(t)x - x}{t} \ \ exists \right\}.$$

Example 3.4. *Consider* $X = C[0,1]$ *with* $\|\cdot\|_\infty$ *norm. The family of operator* $\{T(t)\}_{t \geq 0}$ *define,*

$$T : X \to X \text{ as}$$
$$T(t)f(\xi) = f\left(\tfrac{\xi}{1+t\xi}\right), \xi \in [0,1].$$

Then $T(t)$ *is a* C_0*-semigroup on* X*.*

Let A *be the infinitesimal generator of* $T(t)$*. Then*

$$\begin{aligned} A(f(\xi)) &= \lim_{t \downarrow 0} \frac{T(t)f(\xi) - f(\xi)}{t} \\ &= \lim_{t \downarrow 0} \frac{f\left(\frac{\xi}{1+t\xi}\right) - f(\xi)}{t} \\ &= -\xi^2 \frac{d}{dt} f(\xi). \end{aligned}$$

Definition 3.5. *Consider $T(t)$ be a C_0-semigroup on X. If*

$$\|T(t)\| \leq 1, \forall t \geq 0,$$

Then $T(t)$ is called a C_0-semigroup of contraction.

3.2 m-Accretive operator

Definition 3.6. *[2] Let X be a Banach space and let X^* be its dual. For every $x \in X$ define the duality map J as*

$$J(x) = \{x^* \in X^* : (x^*, x) = \|x\|^2 = \|x^*\|^2\},$$

where (x^*, x) denotes the value of x^* at x.

Definition 3.7. *[2] A nonlinear operator $A : D(A) \subset X \to X$ is called m-accretive if*

$$(Ax - Ay, J(x-y)) \geq 0, \forall x, y \in D(A), \text{ and } R(I + A) = X,$$

where $R(.)$ is the range of an operator.

Lemma 3.8. *[7] If $-A$ is the infinitesimal generator of a C_0 semigroup of contractions then A is m-accretive i.e.*

$$(Au - Av, J(u - v)) \geq 0, \forall u, v \in D(A),$$

where J is the duality map and $R(I + \lambda A) = X$, for $\lambda > 0$, where I is the identity operator on X and $R(.)$ denotes range of an operator.

Lemma 3.9. *[9] Let $-A$ be the infinitesimal generator of a C_0 semigroup of contractions. If $X^n \in D(A)$, $n = 1, 2, 3, \ldots$, $X^n \to u \in H$ and if $\|AX^n\|$ are bounded, then $u \in D(A)$ and $AX^n \to Au$.*

4 Method of Rothe for solving Fractional Integral Diffusion Equation

In this section we discuss the work of [2].

In [2] author apply the Rothe's method to the following fractional integral diffusion equation in a Banach space X.

$$\frac{\partial u(t)}{\partial t} + Au(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{u(s)}{(t-s)^{1-\alpha}} ds + f(t), \ t \in (0,T]. \\ u(0) = u_0. \tag{1}$$

where $0 < \alpha < 1, \ -A$ is the infinitesimal generator of a C_0-semigroup of contractions, f is a given map from $[0,T]$ into $X, \ u_0 \in D(A) \subset A$, the domain of A.

4.1 Assumptions

(A1) There exists a constant $k > 0$ s.t.

$$\|f(t) - f(s)\| \leq k|t-s|, \ \forall t, s \in [0,T].$$

(A2) Suppose that T and α satisfy the following relation

$$\frac{T^{1+\alpha)}}{\Gamma(1+\alpha} < 1.$$

where Γ is the Gamma function.

4.2 Statement of the result

Definition 4.1. *A strong solution u of* (1) *on* $[0,T]$ *is a function* $u \in C([0,T],X)$ *such that* $u(t) \in D(A)$ *for a.e.* $t \in [0,T]$, *u is differentiable a.e. on* $[0,T]$ *and*

$$\frac{\partial u(t)}{\partial t} + Au(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{u(s)}{(t-s)^{1-\alpha}} ds + f(t), \ a.e. \ t \in (0,T]. \\ u(0) = u_0.$$

Theorem 4.2. *Let the assumptions* (A1) *and* (A2) *are satisfied. Then for every* $u_0 \in D(A)$, *the initial value problem* (1) *has a unique strong solution on the interval* $[0, T]$.

4.3 Constructions of Rothe's sequence

To apply the method of semidiscretization, author divided the interval $[0, T]$ into the subintervals of length $h_n = \frac{T}{n}$. Then replace the equations (1) by the following approximate equations.

$$\frac{u_1^n - u_0^n}{h_n} + Au_1 = f_0,$$
$$u_0^n = u_0,$$

and for $j = 2, 3, 4, \ldots, n,$,

$$\frac{u_j^n - u_{j-1}^n}{h_n} + Au_j^n = \frac{1}{\Gamma(1+\alpha)} \sum_{i=1}^{j-1} u_i[(t_j^n - t_{i-1}^n)^\alpha - (t_j^n - t_i^n)^\alpha] + f_{j-1}^n,$$

where, $f_j^n = f(t_j^n)$

Next define the Rothe's sequence $\{U^n\}$ as,

$$U^n(t) = \begin{cases} u_0 & if \quad t = 0, \\ u_{j-1}^n + \frac{1}{h_n}(t - t_{j-1}^n)(u_j^n - u_{j-1}^n) & if \quad t \in (t_{j-1}^n, t_j^n]. \end{cases}$$

After that, author proved that $\{U^n\}$ converges to the solution of the considered problem as $n \to \infty$ using some apriori estimates on u_j^n and $\frac{u_j^n - u_{j-1}^n}{h_n}$. In the end, they have established that the solution is a strong solution and it is unique.

4.4 Example

As an application of the problem (1) author consider the following example

$$\begin{aligned} \frac{\partial u(t,x)}{\partial t} - \frac{\partial^2 u(t,x)}{\partial x^2} &= \frac{1}{\Gamma(1/2)} \int_0^t \frac{u(s,x)}{(t-s)^{\frac{1}{2}}} ds \qquad \text{on } [0,1]\times[0,\pi], \\ u(0,x) &= u_0(x), \\ u(t,0) = u(t,\pi) &= 0 \qquad \forall t \in [0,1], \end{aligned} \tag{2}$$

where $u : [0,1]\times[0,\pi] \to \mathbb{R}$ is an unknown function and $u_0 : [0,\pi] \to \mathbb{R}$ is a given initial value function.

Identify $u : [0,1] \to L^2([0,\pi])$ by $u(t)(x) = u(t,x)$, and define

$Au = -\frac{\partial^2 u}{\partial x^2}$, and

$D(A) = \{u \in L^2([0,\pi]) | u'' \in L^2([0,\pi])\}$,
then the problem (2) reduces to,

$$\begin{aligned} \frac{\partial u(t)}{\partial t} + Au(t) &= \frac{1}{\Gamma(1/2)} \int_0^t \frac{u(s)}{(t-s)^{\frac{1}{2}}} ds \qquad t \in [0,T], \\ u(0) &= u_0, \end{aligned}$$

which is same as problem (1).

Here, $f(t) = 0$, $T = 1$ and $\alpha = 1$.
So,

$$\frac{T^{1+\alpha}}{\Gamma(1+\alpha)} = \frac{1}{\Gamma(\frac{3}{2})} < 1.$$

Hence the condition (A1) and (A2) are satisfied.
So by applying the Theorem (4.2) obtain a unique strong solution of the given problem.

4.5 Observation

The condition (A2) which is mentioned in [2], it means that [2] provides only a local (in time) unique strong solution of (1). For example, by using main results of [2] the following fractional integral diffusion equation problem cannot be solved

$$\frac{\partial u(t,x)}{\partial t} - \frac{\partial^2 u(t,x)}{\partial x^2} = \frac{1}{\Gamma(0.5)} \int_0^t \frac{u(s,x)}{(t-s)^{0.5}} ds \quad \text{in } (0,10)\times(0,\pi),$$
$$u(0,x) = u_0(x), \quad \forall x \in [0,\pi],$$
$$u(t,0) = u(t,\pi) = 0 \quad \forall t \in [0,10].$$

Since $\frac{10^{1.5}}{\Gamma(1.5)} > 1$ and the smallness condition (A2) is not satisfied. So we can not apply the Theorem (4.2).

5 Method of Rothe for solving Fractional Differential Equation

In this section we discuss the work of [4].

In [4] author apply the Rothe's method to the following semilinear fractional differential equation in a Banach space X.

$$\begin{aligned} D^\alpha u(t) + Au(t) &= f(t,u(t)),\ t \in (0,T], \\ u(0) &= u_0, \end{aligned} \tag{3}$$

where, $D^\alpha (0 < \alpha < 1)$ denotes the standard Riemann-Liouville fractional derivative of order α, $-A : D(A) \subset X \to X$ is the infinitesimal generator of a C_0-semigroup $S(t),\ t \geq 0$ of contractions in $X,\ u_0 \in D(A)$, the domain of A, and the map $f : I \times D(A) \to X$ is continuous. Here $I = [0,T]$.

5.1 Assumptions

(H1) There exists a constant $k_1 > 0$ s.t.

$$\|f(t,u) - f(s,v)\| \leq k_1[|t-s| + \|u-v\|],\ \forall t,s \in [0,T],\ \forall u,v \in D(A).$$

5.2 Statement of the result

Theorem 5.1. *Suppose that* (H1) *is satisfied and* A *is* m*-accretive. Then problem* (3) *has a unique strong solution on* I.

5.3 Constructions of Rothe's sequence

To apply the method of time discretization, author divided the time interval $[0, T]$ into the subintervals of length $h_n = \frac{T}{n}$ and replace the equations (3) by the following approximate equations.

$$\frac{u_j^n - \alpha u_{j-1}^n}{h_n^\alpha} + Au_j^n = f(t_j^n, u_{j-1}^n), \; j = 1,2,\ldots,n,$$
$$u_0^n = u_0.$$

Next define the Rothe's sequence $\{U^n\}$ as,

$$U^n(t)= \begin{cases} u_0 & for \quad t \in [-\tau, 0], \\ u_{j-1}^n + \frac{1}{h_n^\alpha}(t - t_{j-1}^n)(u_j^n - \alpha u_{j-1}^n) & in \quad I_j^n = (t_{j-1}^n, t_j^n], \\ & j = 1,2,\ldots,n. \end{cases}$$

After that, author proved that $\{U^n\}$ converges to the solution of the considered problem as $n \to \infty$. For this, first proved some apriori estimates on u_j^n and $\frac{u_j^n - \alpha u_{j-1}^n}{h_n^\alpha}$ using (H1). In the end, they have established that the solution is a strong solution and it is unique.

6 Conclusion

The work [2] and [4] can be extended for further study on this field. As example we can study this type of equation by taking some non-local condition. Also we can study the use of Rothe's Method to establish the existence and uniqueness of a strong solution for a fractional neutral functional differential equation, as these types of equations have numerous applications in the field of science and technology.

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Flow around a square cylinder at low Reynolds numbers

Dharmaraj Deka
Department of Mathematical Sciences, Tezpur University, Tezpur 784028, India

email: dharma@tezu.ernet.in

Abstract. This article pertains to the solution of the well-known flow past a square cylinder problem. The problem is governed by two dimensional (2D) transient Navier-stokes (N-S) equation. A very recent higher order compact scheme has been employed to discretize the N-S equation on nonuniform grid. The scheme adopted here shows at least third order of spatial convergence and second order of temporal convergence. Focus of the article is laid on attaining the unsteady periodic solution of the flow problem considered. Numerical simulations are carried out for $Re = 60, 100, 200$. The computed data are presented graphically or in tabulated form along with the results available in the literature.

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Keywords. Square cylinder, Navier-Stokes equation, Nonuniform grid, Higher-order compact schemes.

1 Nomenclature

B	blockage ration	u	component of velocity in the $x-$direction
C_D	drag coefficient	U_∞	characteristic velocity
C_L	lift coefficient	v	component of velocity in the $y-$direction
g	acceleration due to gravity	λ	clustering parameter
L	characteristic length	ν	kinematic viscosity
Re	Reynolds number	ϕ	unknown transport variable
St	Strouhal number	ψ	streamfunction
t	nondimensional time	ω	vorticity
$\mathbf{u}$	velocity vector		

2 Introduction

The streamfunction-vorticity ($\psi-\omega$) formulation of N-S equation in non-dimensional form can be written as,

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=-\omega, \tag{4}$$

$$u\frac{\partial\omega}{\partial x}+v\frac{\partial\omega}{\partial y}=\frac{1}{Re}\left(\frac{\partial^2\omega}{\partial x^2}+\frac{\partial^2\omega}{\partial y^2}\right). \tag{5}$$

where $u=\psi_y$ and $v=-\psi_x$. The N-S equation governs the fluid flow of various critical configurations found in the field of fluid dynamics. One such problem is the flow past square cylinder problem. Owning to its theoretical significance and practical relevance, recent years have seen noteworthy interest laid on to this problem. The importance of this problem is due to their multifaceted flow configuration that depends on the Re. This lead to various theoretical and experimental investigations of flow past square cylinder.

The confined free flow around bluff bodies, especially cylinders with circular and square cross-section, has been investigated in detail by many researchers for a very long time. These fluid-structure interaction is of practical importance in many fields of engineering such as designing bridges, building and offshore structures. The availability of open literature about flow past a circular cylinder is overwhelming, whereas the analogous case of the flow past square cylinder has got much lesser attention [1, 2, 3, 4]. Nevertheless, it is well documented in the literature that accurate simulation around bluff bodies require body fitted orthogonal grids with clustering on the surface. In the context of finite difference approximation it entails coordinate transformation to generate grid around circular as well as square cylinder. Our formulation alleviates such requirement and as discussed earlier should be free from destabilizing effects as noted earlier [5]. To the best of our knowledge high order transformation free FD computation of flow past square cylinder is not available in the literature.

In this work, our focus is on simulating the flow for different values of *Re*. The following section addresses problem details and the computed results of the present investigation.

3 Results and discussion

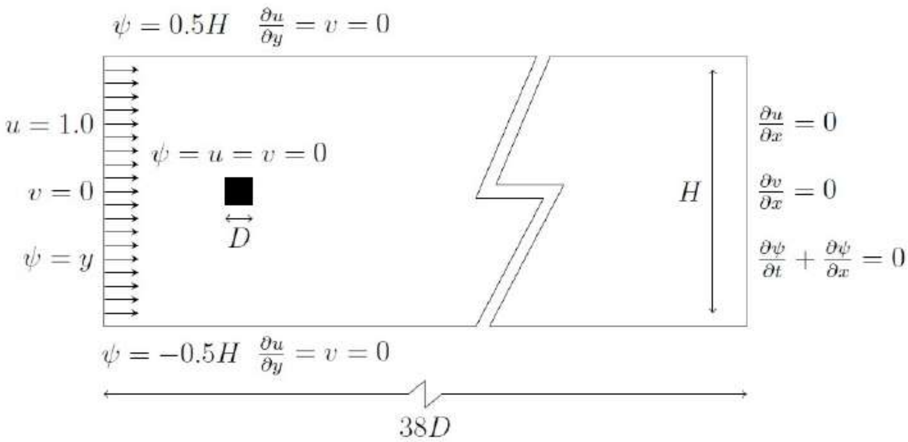

Fig. 1: Problem 2: Schematic diagram of the configuration for flow past a square cylinder problem.

The solution procedure is now applied to unsteady flow past a stationary square cylinder at zero incidence. The numerical setup for the flow configuration has been presented in Fig. 1. A 2D stationary square with side length $D = 1$ is placed in the

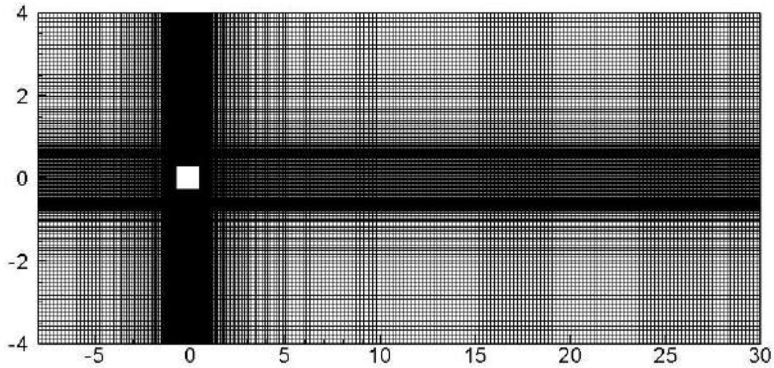

Fig. 2: Problem 2: Nonuniform grid generated using trigonometric stretching functions.

computation domain in a fashion such that the origin of the Cartesian plane coincides with the centre of the square. The cylinder is exposed to a freestream velocity $u_\infty = 1$. The top and bottom sidewalls of the computational domain are considered to be equidistant from the centre of the cylinder with a distance between them kept fixed at H. For this problem two values of H have been considered *viz.* $H = 8D$. This results in blockages $B = 0.125$. In order to reduce the effects of inflow and outflow boundary conditions on the flow field the distances of upstream and downstream boundaries for the centre of the cylinder are considered to $L_u = 8D$ and $L_d = 30D$. The Reynolds number, defined as $Re = u_\infty D/\nu$, depends on the cylinder width and the freestream velocity with ν being the kinematic viscosity. Other boundary conditions are as shown in Fig. 2. In this study we cluster grids on the surface of the cylinder Fig. 2.

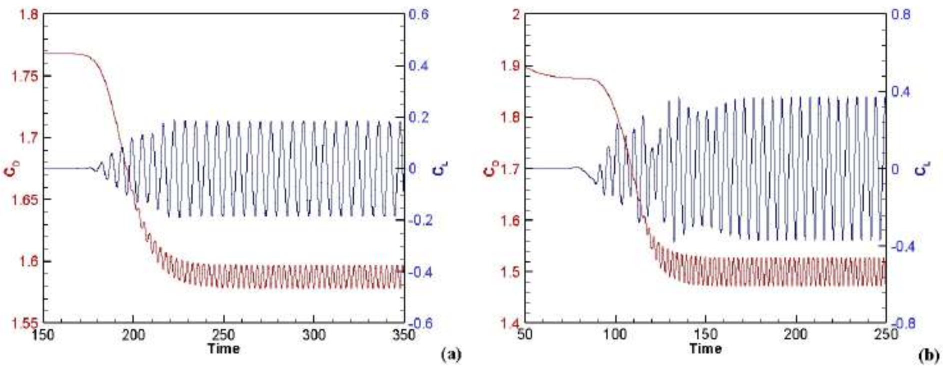

Fig. 3: Problem 2: Time evolution of C_D(red) and C_L(blue) for (a) $Re = 100$ and (b) $Re = 200$.

In the highly accurate work done by Breuer et al. [2], the authors established that

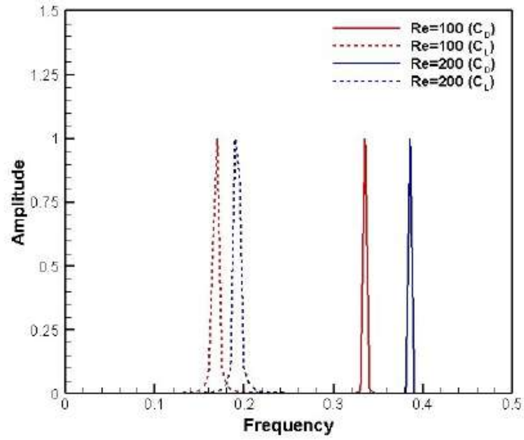

Fig. 4: Problem 2: Power spectra of C_D and C_L for $Re = 100$ and 200.

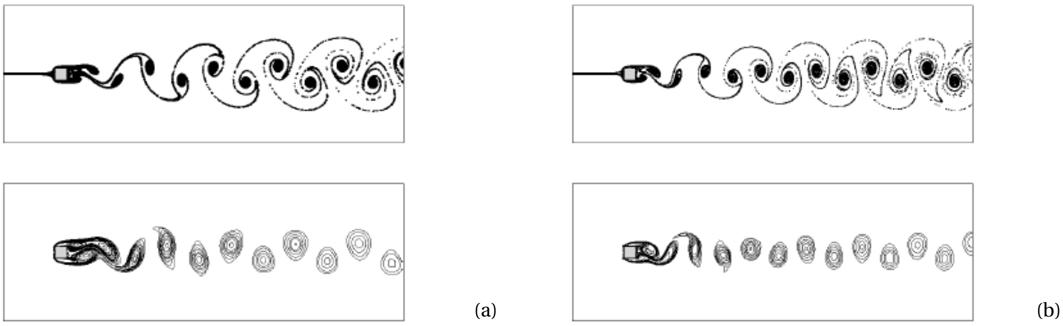

Fig. 5: Problem 2: Streaklines (top) and vorticity (bottom) contours at stable periodic stage for (a) $Re = 100$ and (b) $Re = 200$.

the flow converges to an unsteady periodic state for $Re \geq 60$ using a fine 561×341 grid. Following their work, the present computations are carried out for $Re = 100$ and 200 on a much coarser grid of size 283×141 in order to investigate how efficiently the present scheme can detect unstable periodic solution beyond the bifurcation point. The time step in both the cases are taken to be $1.0e-07$. The present simulation requires large number of nodes in the neighbourhood of the cylinder. A nonuniform grid is laid out in the computational domain as shown in Fig. 2. Wider grid spacing has been used in the downstream region as we strive to capture the Kármán vortex shedding phenomena using lesser number of grid points.

Fig. 3 depicts the time history of drag coefficient (C_D) and lift coefficient (C_L) for the Re values 100 and 200 respectively. From the figures it is clear that the present scheme developed can capture the unsteady periodic flow with ease. To further ascertain the periodic nature of the flow field spectral density analysis has been performed and shown in Fig. 4. The relevant flow parameters *viz.* Strouhal number (St), drag coefficient (C_D) , RMS value of lift coefficient ($\overline{C_L}$) and vortex shedding frequencies (f) of this problem are compiled in Table 1. No experimental and other numerical data with this exact numerical setup is not available in the literature.

Table 1: Problem 2: Values of C_D, C_L and St for $Re = 100$ and 200.

B	Re	St	C_D	$\overline{C_L}$	f
0.1250	100	0.169	1.588	0.130	5.917
	200	0.190	1.500	0.245	5.263

While advancing to the unsteady periodic von Kármán vortex shedding, the flow undergoes several phases. Once vortex shedding is initiated, vortices start shedding in a regular fashion alternatively from both the rare sides of the cylinder. As time progresses, the vortices shed more rapidly, till it meets the desired vortex shedding frequency. The shedding becomes more frequent as the value of Re increases. These features become evident from the data shown in Table 1. These can further be corroborated from Fig. 5, which displays the streaklines and corresponding vorticity contours for both the Re values being considered.

4 Conclusion

Numerical simulation of the classical flow past a square cylinder problem is carried out on nonuniform grid without any coordinate transformations. Unsteady periodic solution for flow past square cylinder is reported for $Re = 100$ and 200. The continuous time evolution of the flow finally leads to the von Kármań vortex shedding phenomena which is clear from the streaklines and vorticity contours presented. Further, the results computed by in the present study are compared to those found in the literature and the present values show a good agreement with the values of previous well established investigations.

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Representation of a number as sums of various polygonal numbers

Satyajit Gayan
Department of Mathematical Sciences, Tezpur University, Sonitpur, Assam, India

email: satyajitgayan1993@gmail.com

Abstract. Jacobi first investigated the number of ways a positive integer can be written as a sum of two squares and found that

$$r_{\{\square+\square\}}(n) = 4(d_{1,4}(n) - d_{3,4}(n)),$$

where $r_{\{\square+\square\}}(n)$ denotes the number of representation of n as a sum of two squares and $d_{i,j}(n)$ is the number of positive divisors of n congruent to i modulo j. Following his steps many mathematicians formulate a number of similar results which involves various polygonal numbers. In this chapter we summarize the works done so far on such representation identities.

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Keywords. Ramanujan's theta function, Polygonal numbers.

1 Introduction

A polygonal number is a type of figurate number that is a generalization of triangular, square, etc., to an n-gon for n an arbitrary positive integer. p. Starting with the n^{th} triangular number T_n, then

$$n+T_{n-1}=T_n.$$

Adding again, we have

$$n+2T_{n-1}=n^2=S_n,$$

which gives the n^{th} square number. Similarly,

$$n+3T_{n-1}=\frac{1}{2}n(3n-1)=P_n,$$

gives the n^{th} pentagonal number. Proceeding the same way r times we will get the n^{th} r-gonal number given by

$$p_n^r=\frac{1}{2}n[(n-1)r-2(n-2)=\frac{1}{2}n[(r-2)n-(r-4).$$

Till date, there have been numerous identities that counts the number of ways a number can be represented as a sum of two or more than two polygonal numbers. The identities can also be used to discard the possibility any such representation. For example 5 cannot be written as a sum of two triangular numbers.

The main tools used to derive such identities are nothing but the Ramanujan's theta function $f(a,b)$ and its special cases $\varphi(q)$ and $\psi(q)$ which are given below.

$$f(a,b):=\sum_{n=-\infty}^{\infty}a^{n(n+1)/2}b^{n(n-1)/2}, \qquad |ab|<1,$$

$$\phi(q):=f(q,q)=\sum_{n=-\infty}^{\infty}q^{n^2},$$

$$\psi(q):=f(q,q^3)=\sum_{n=0}^{\infty}q^{\frac{n(n+1)}{2}}.$$

2 Representation of a number as sums of various polygonal numbers

Jacobi's celebrated two square theorem is

Theorem 2.1. *The number of representations of a positive integer n as a sum of two squares is*

$$r_{\{\square+\square\}}(n)=4(d_{1,4}(n)-d_{3,4}(n)),$$

where $r_{\{\square+\square\}}(n)$ denotes the number of representation of n as a sum of of two squares and $d_{i,j}(n)$ is the number of positive divisors of n congruent to i modulo j.

Similar representation theorem found by Dirchlet, Lorenz, Legendre and Ramanujan [2] . For example Lorenz [2] fount that

Theorem 2.2. *The number of representations of a positive integer n as a sum of a squares and three times a square is*

$$r_{\{\square+3\square\}}(n) = 2(d_{1,3}(n) - d_{2,3}(n)) + 4(d_{4,12}(n) - d_{8,12}(n)).$$

Hirschhorn [2], in 2003 proved similar results, which involves multiples of squares and triangular numbers.

Theorem 2.3. *For* $n \geq 0$,

$$\begin{aligned}
r_{\{\triangle+\triangle\}}(n) &= d_{1,4}(4n+1) - d_{3,4}(4n+1),\\
r_{\{\square+2\triangle\}}(n) &= d_{1,4}(4n+1) - d_{3,4}(4n+1),\\
r_{\{2\square+\square\}}(n) &= d_{1,4}(8n+1) - d_{3,4}(8n+1),\\
r_{\{\triangle+4\triangle\}}(n) &= \frac{1}{2}(d_{1,4}(8n+5) - d_{3,4}(8n+5)),\\
r_{\{\triangle+12\triangle\}}(n) &= \frac{1}{2}(d_{1,3}(8n+13) - d_{2,3}(8n+13)).
\end{aligned}$$

Hirschhorn [2], also proved twenty-nine representation theorems involving triangular numbers, squares, pentagonal numbers and octagonal numbers. Below we list some of them.

Theorem 2.4. *For* $n \geq 0$,

$$\begin{aligned}
r_{\{\triangle+\triangle\}}(n) &= d_{1,6}(6n+1) - d_{5,6}(6n+1),\\
r_{\{\square+4\pi\}}(n) &= d_{1,24}(24n+7) + d_{19,24}(24n+7) - d_{5,24}(24n+7) - d_{23,24}(24n+7),\\
r_{\{\triangle+2\Omega\}}(n) &= d_{1,24}(24n+19) + d_{19,24}(24n+19) - d_{5,24}(24n+19) - d_{23,24}(24n+19),\\
r_{\{2\square+\pi\}}(n) &= d_{1,3}(24n+1) - d_{2,3}(24n+1),
\end{aligned}$$

where π and Ω denotes pentagonal and octagonal numbers respectively.

In 2007 Lam [5] presented eighteen infinite products and their Lambert series expansions that involves the Ramanujan's theya functions $\varphi(q)$ and $\psi(q)$. From the obtained results, he deduce a number of representation results for the number of representations of an integer n by eighteen quadratic forms in terms of divisor sums. He proved the following results. In this chapter we list some of them.

Theorem 2.5.

$$\varphi(q)\varphi(q^4) = 1+\sum_{j=1}^{\infty}\frac{(-1)^j q^{2j}}{1+q^{4j}} - 2\sum_{j=1}^{\infty}\frac{(-1)^j q^{2j-1}}{1-q^{2j-1}},$$

$$\varphi(q)\psi(q^8) = -\sum_{j=1}^{\infty}\frac{(-1)^j q^{2j-2}}{1-q^{2j-1}} - \sum_{j=1}^{\infty}\frac{(-1)^j q^{2j-1}}{1-q^{4j}},$$

$$\varphi(q^4)\psi(q^2) = -\frac{1}{2}\sum_{j=1}^{\infty}\frac{(-1)^j q^{\frac{j-1}{2}}}{1-q^{\frac{2j-1}{2}}} - \frac{1}{2}\sum_{j=1}^{\infty}\frac{(-1)^j (-q)^{\frac{j-1}{2}}}{1-(-q)^{\frac{2j-1}{2}}}.$$

Theorem 2.6.

$$\varphi^3(q)\psi(q^8) = 2\sum_{j=1}^{\infty}\frac{(-1)^j q^{2j}}{1+q^{4j}} - 2\sum_{j=1}^{\infty}\frac{(-1)^j q^{2j-1}}{1-q^{2j-1}},$$

$$\varphi^2(q)\psi^2(q) = -\sum_{j=1}^{\infty}\frac{j q^{j-2}}{1+(-q)^j} - 2\sum_{j=1}^{\infty}\frac{(-1)^j j q^{4j-1}}{1+q^{4j}} + \sum_{j=1}^{\infty}\frac{(-)^j (2j-1) q^{2j-2}}{1+q^{4j-2}},$$

Theorem 2.7.

$$\varphi^4(q)\varphi^2(q^2) = 2\sum_{j=1}^{\infty}\frac{(-1)^j q^{2j}}{1+q^{4j}} - 2\sum_{j=1}^{\infty}\frac{(-1)^j q^{2j-1}}{1-q^{2j-1}},$$

$$\varphi^2(q)\psi^2(q) = -\sum_{j=1}^{\infty}\frac{j q^{j-2}}{1+(-q)^j} - 2\sum_{j=1}^{\infty}\frac{(-1)^j j q^{4j-1}}{1+q^{4j}} + \sum_{j=1}^{\infty}\frac{(-)^j (2j-1) q^{2j-2}}{1+q^{4j-2}},$$

$$\psi^4(q)\psi^2(q^2) = -\sum_{j=1}^{\infty}\frac{j^2 q^{j-1}}{1+(-q)^{2j-1}},$$

$$\varphi^2(q)\psi^4(q^2) = -\sum_{j=1}^{\infty}\frac{(2j-1)^2 q^{j-1}}{1+(-q)^{2j-1}}.$$

Some of the results are given by S. Ramanujan and some of them proved by S.H. Shan. Using the above identities he proved the following representation result that involves squares and triangular numbers.

Corollary 2.8. *For $n \geq 1$,*

$$r(\square+\square+\square+8\triangle) = k(n) \sum_{d|n+1, d\ odd} d,$$

where

$$k(n)=\begin{cases} 6 & : n \equiv 1\ (mod\ 4), \\ 3 & : n \equiv 2\ (mod\ 4), \\ 8 & : n \equiv 3\ (mod\ 8), \\ 1 & : n \equiv 0\ (mod\ 4), \\ 0 & : n \equiv 7\ (mod\ 8). \end{cases}$$

In 2012 Baruah and Sarmah [1] present sets of identities involving *decagonal* numbers, *hendecagonal* numbers, *dodecagonal* numbers, *heptagonal* numbers and *octadecagonal* numbers. As elementary tools they used the dissection of $\phi(q)$, $\psi(q)$ and $G_k(q)$, where

$$G_k(q) := f(q, q^{k-3}).$$

In this chapter we list some of their results.

- **Identities involving decagonal numbers**.

Theorem 2.9.

$$\begin{aligned} r(\square+3F_{10})(n) &= d_{1,3}(16n+27) - d_{2,3}(16n+27), \qquad (6) \\ r(2\triangle+3F_{10})(n) &= \frac{1}{2}(d_{1,3}(16n+31) - d_{2,3}(16n+31)), \\ r(2\triangle+F_{10})(n) &= \frac{1}{2}(d_{1,4}(16n+13) - d_{3,4}(16n+13)), \\ r(6\triangle+F_{10})(n) &= \frac{1}{2}(d_{1,3}(16n+21) - d_{2,3}(16n+21)), \\ r(3\triangle+3F_{10})(n) &= d_{1,3}(16n+9) - d_{2,3}(16n+9). \end{aligned}$$

- **Identities involving hendecagonal numbers**.

Theorem 2.10.

$$r(\triangle+F_{11})(n)=d_{1,12}(36n+28)-d_{11,12}(6n+29), \quad (7)$$

$$r(\triangle+2F_{11})(n)=d_{1,8}(72n+107)-d_{7,8}(72n+107),$$

$$r(F_{10}+F_{11})(n)=d_{1,8}(144n+179)-d_{7,8}(144n+179),$$

$$r(2\triangle+F_{11})(n)=d_{1,8}(72n+67)-d_{7,8}(72n+67),$$

$$r(\square+4F_{11})(n)=d_{1,8}(18n+49)+d_{3,8}(18n+49)-d_{5,8}(18n+49)-d_{7,8}(18n+49).$$

• **Identities involving dodecagonal numbers**.

Theorem 2.11.

$$r(5\square+F_{12})(n)=d_{1,4}(5n+4)-d_{3,4}(5n+4), \quad (8)$$

$$r(F_{12}+F_{12})(n)=d_{1,4}(5n+8)-d_{3,4}(5n+8),$$

$$r(5\triangle+F_{12})(n)=\frac{1}{2}(d_{1,4}(20n+17)-d_{3,4}(20n+17)).$$

• **Identities involving octadecagonal numbers**.

Theorem 2.12.

$$r(F_5+F_{18})(n)=d_{1,24}(96n+151)+d_{19,24}(96n+151)-d_{5,24}(96n+151)-d_{23,24}(96n+151), \quad (9)$$

$$r(\triangle+F_{18})(n)=\frac{1}{2}(d_{1,4}(32n+53)-d_{3,4}(32n+53)),$$

$$r(3\triangle+F_{18})(n)=\frac{1}{2}(d_{1,}(32n+61)-d_{2,}(32n+61)).$$

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